Hypercomplex Iterations

Distance Estimation and Higher Dimensional Fractals

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Hypercomplex Iterations

Distance Estimation and Higher Dimensional Fractals

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To Yumei's father, Ge Dang, and to the memory of Lou's mother,
Alice Fisher Kauffman

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Preface

This book is an exploration of methods to visualize fractals in spaces of dimension two, three, and higher. Our basic method involves estimating the distance to a fractal that is generated through the iteration of a complex or hypercomplex function of many variables. The book is based on Yumei Dang's doctoral dissertation and on previous work in fractal geometry, including the work of John Hart, Louis Kauffman, and Dan Sandin.

Julia sets of quadratic functions, as well as many other deterministic fractals, exist in spaces of higher dimension than the complex plane. Visualization of hypercomplex Julia sets is much more difficult than visualization in the complex plane, and the efficiency of the algorithms becomes a significant issue. Several algorithms have been used to view these structures that either required a large amount of storage space or used a low resolution rendering method. One ray-tracing algorithm using distance estimation was given in which the estimation for quaternion space works well, but the algorithm was not justified mathematically. During our study and research, a mathematical justification of this algorithm in both the original complex case and its higher dimensional analogues was discovered.

The distance estimation formula provides an approximation of the distance from a point outside of a Julia set to the Julia set itself. Since Julia sets are fractals, it is extremely difficult to get an exact evaluation of the distance from a point to a Julia set. Therefore, the lower bounds and upper bounds become very important for the purpose of visualization. The distance estimate is a delicate matter

to establish even in the case of Julia sets over the complex numbers. It depends on the Riemann Mapping Theorem and a number of special results in complex analysis. It has been verified empirically that in most cases the exact analogue of this distance estimate works very well for quaternionic Julia and Mandelbrot sets in three and four dimensions. The justification of this experimental work demands both new theoretical work and a more fine-grained approach to the experimentation.

On the theoretical side, we pursued a program to analyze the distance estimate using theorems in analysis over the quaternions. Then, we found a mathematical justification and description for the distance estimate in higher dimensional spaces by using a geometric method. This new method provides a fundamental theoretical background for the visualization of deterministic fractals.

To complement this theoretical base, we pursue optimization techniques to improve the distance estimation algorithm and corresponding rendering methods on powerful computer systems. Since images of hypercomplex fractals are time consuming to create, the flexibility of the experiments is limited. This is the primary mathematical reason for wanting the best supercomputers to help in this problem. In more than two years of experiments on the Alpha-Cluster at PSC, the SGI Power Challenge at NCSA and the virtual reality environment CAVE at EVL, finer and more flexible results have been obtained.

We expect that a high-quality and flexible source of images of higher dimensional Julia sets, Mandelbrot sets and other deterministic fractals will raise many new mathematical questions.

The book is organized in four parts. In Part I, Chapter 1 gives an introduction to hypercomplex iterations and Chapter 2 discusses deterministic fractals and distance estimation.

Part II, covered by Chapters 3 through 5, provides mathematical background on the classical method for estimating the distance to complex fractals. Chapter 4 provides the reader with a wealth of interesting lore about quaternionic analysis, though its eventual conclusion is pessimistic about the application of these techniques to hypercomplex distance estimation. Chapter 5 is an interlude about the relationships among quaternions, topology, and quantum mechanics.

Part III begins the serious analysis of hypercomplex distance estimation. In Chapter 6, we show how to estimate the distance to quaternion Mandelbrot sets by reducing the problem to the complex case. In Chapter 7 we give our new methods for distance estimation, outlined in Chapter 1, and apply them to specific cases of higher dimensional iterations.

Part IV discusses in detail issues involved in generating, modelling and rendering hypercomplex fractals. Chapters 8 and 9 present two different algorithms for modelling deterministic fractals: inverse iteration and ray-tracing by distance estimation. A brief description of other visualization methods (including animation) on platforms such as the CAVE and ImmersaDesk, supercomputing techniques, and new data structures are described in Chapter 10.

This book was written using the $\text{\LaTeX}$ document processing system. The 2D illustrations were created with Silicon Graphics IRIS-4D workstations. The 3D illustrations were created on SGI Power-Challenge and IRIS-4D workstations.