EVOLUTION EQUATIONS AND APPROXIMATIONS

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Preface

Abstract evolution equations provide a unifying framework in order to investigate well-posedness of dynamical systems of various types describing the time evolution of concrete systems. Moreover, the abstract setting can also be used as a framework in order to develop approximation schemes for such dynamical systems. In case of linear autonomous problems the abstract theory is available in a considerable number of textbooks on strongly continuous semigroups of linear bounded operators. Also the cases of linear time dependent problems and of nonlinear autonomous problems are already well represented by monographs. One of the main difficulties arising in connection with nonlinear problems is the fact that the usual requirements of dissipativity resp. accretiveness on the operators governing the evolution equation result in too severe restrictions in case of the concrete problems one wants to consider. The reason is that dissipativity of an operator is global property. A sufficiently general and flexible way for localizing the concept of dissipativity has been developed by K. Kobayashi, Y. Kobayashi and S. Ôharu by requiring dissipativity of the operators only on level sets of lower semi-continuous functionals. The common methodological feature of the proofs for existence resp. well-posedness is to prove convergence of the implicit (or backward) Euler scheme on certain time mesh sequences (i.e., to prove convergence of what is usually called a DS-approximation), which leads to the concept of mild solutions. Uniqueness proofs in many cases are based on the concept of integral solutions introduced by Ph. Benilan.

This monograph intends to give a unified presentation of the general approach for well-posedness results, the core being the theory developed by K. and Y. Kobayashi together with S. Ôharu contained in Sections 6.1 – 6.5. Modifications of this theory are introduced for quasilinear problems in Section 6.9 and in Chapter 7, where a localized version of the theory developed by M. G. Crandall and A. Pazy is presented. In Section 6.10 we show that this approach can be used for nonlinear parabolic problems, where the operator governing the evolution equation is the subdifferential of a convex and lower semi-continuous functional. In Chapter 8 apply the difference scheme approach to variational formulations of the evolution equation using the concept of Gelfand triples as the basic setting. The only exception where well-posedness is not established by a difference scheme approach is the class of semilinear evolution equations considered in Chapter 11. There we use a Banach fix point argument instead.

For the sake of completeness of the representation we include in Chapter 2 a short introduction into the theory of C_0 -semigroups of bounded linear operators restricting ourselves to a discussion of strong and mild solutions for abstract Cauchy problems, to the Hille–Yosida theorem, the Lumer–Phillips theorem and Ball’s theorem concerning characterization of well-posedness via weak solutions. Because of their importance we give in Chapter 3 the basic facts on analytic semigroups including the characterization of infinitesimal generators as sectorial operators. As a first step to the general nonlinear and time dependent case we discuss in Chapter 5 strongly continuous semigroups of nonlinear Lipschitzian operators. We present the Crandall–Liggett generation theory and give a detailed discussion of various solution concepts for nonlinear evolution equations including the concept of integral solutions (as introduced by Ph. Bénilan). We discuss also carefully the concepts of weak and strong generators as well as their relation to the minimal sections of the operator generating the semigroup and present finally a Hille–Yosida type theorem for nonlinear semigroups.

The second central theme for this monograph are abstract approximation results for evolution equations. In Chapter 4 we present different versions of the basic approximation result for C_0 -semigroups of linear bounded operators known as Trotter–Kato theorem. Section 8.2 contains an approximation result of Lax–Richtmyer type (i.e., stability plus consistency imply convergence) for the evolution equations considered in Chapter 8. Approximation of mild solutions of evolution equations in the framework of Chapter 6 is the topic of Chapter 10. An abstract approximation result of Lax–Richtmyer type is given in Section 10.1, whereas a Chernoff type approximation result is obtained in Section 10.2. As an application of the general result given in Section 10.1 we state and prove a Trotter–Kato theorem for nonlinear semigroups. The general Chernoff type theorem is used to prove some results on operator splitting in Section 10.3. Finally, Chapter 11 contains approximation results for the semilinear evolution equations considered there (see Sections 11.3 and 11.4).

In order to demonstrate the feasibility of the existence and uniqueness theory presented in this monograph for various types of evolution equations we include applications to more or less concrete problems. As an application of the theory for nonlinear semigroups presented in Chapter 5 (and the results on subdifferentials of lower semi-continuous convex functionals given in Section 1.6) we consider nonlinear diffusion in Section 5.4. The main applications are those of Chapter 9. In Section 9.1 delay equations of various type are considered. Well-posedness for scalar conservation laws of general type is discussed extensively in Section 9.2. Our approach is based on the viscosity method and applies the general theory developed in Chapter 6 to the viscous problems, whereas the limiting problem is handled with the localized version

of the Crandall–Pazy theory as presented in Chapter 7. We show that the mild solution obtained via the Crandall–Pazy theory is the unique entropy solution of the problem. Finally, in Section 9.3 we apply the general theory to the Navier–Stokes equations of dimension two and three.

The presentation of the material in this book is naturally influenced by many sources. Besides the other papers and books quoted at various parts, we in particular have to mention the books by V. Barbu, I. Miyadera, N. H. Pavel and A. Pazy (see [Bb2], [Bb4], [Miy], [Pav2] and [Pa1]). In general we did not try to follow the history of the results presented in this book in detail and to give a complete survey on the relevant literature. This means also that we do not claim authorship for results for which there is no quotation of references. However, we believe that the book contains several components which are not available in the presently existing literature. Specifically we want to mention the following items: the statement and proof of Chernoff’s formula for rather general time dependent evolution equations (see Theorem 10.16) in Section 10.2; the treatment of scalar conservation laws in Section 9.2 combining the semigroup approach by M. G. Crandall and the more direct one based on the concept of entropy solutions by S. N. Kružkov (see [Cr2] and [Kr]); the ‘localized’ version of the Crandall–Pazy theory for evolution equations in Chapter 7; the results on delay equations of various types presented in Section 9.1 and the treatment of the Navier–Stokes equations in the three dimensional case in Subsection 9.3.2.

We tried to give a self-contained presentation of the material as far as possible. In order to serve this purpose we present in the introductory Chapter 1 the results (usually with complete proofs) on dissipative operators, monotone operators and the Minty–Browder theory together with the basic fact on sub-differentials of convex functionals. In the Appendix we collect some of the technical results which are needed in other parts of the book. Whenever we had to quote results from the literature we tried to give precise references.

It may be helpful for the reader who wants to give a special course or a seminar on one of the topics of this monograph to have some information on the mutual dependence of the various chapters:

a) Chapter 2 can serve as an introduction to the theory of C_0 -semigroups of bounded linear operators. In order to include the Lumer–Phillips theorem one has to use the results of Section 1.2 specialized to linear operators up to Theorem 1.10. In addition one would need the definition of the duality mapping and the material from Definition 1.2 up to Lemma 1.3. In order to deal with the example of Section 2.5 the definition of Gelfand triples and the Lax–Milgram theorem should be taken from Section 3.1. If one adds Chapter 3 to Chapter 2 then one could include the basic fact about analytic semigroups into an introduction to C_0 -semigroups. If one wants to include also approximation

then Chapter 4 or parts of it (for instance Section 4.1 up to Proposition 4.3 and Section 4.2) can be added.

b) Sections 5.1 – 5.3 provide a rather self-contained introduction to the theory of strongly continuous semigroups of nonlinear Lipschitzian operators. Of course one has to include most of the material covered in Sections 1.1 – 1.4. If one wants to include the evolution equation describing nonlinear diffusion of Section 5.4 one needs also material from Sections 1.5 and 1.6.

c) If one intends to give an introduction to the Kobayashi–Oharu theory, then Sections 6.1 – 6.5 and 6.7 provide a rather self-contained basis. One could also include Section 6.9. As an application one can add without further requirements the results on delay equations given in Section 9.1. For the material of Sections 6.6 and 6.8 we need in addition results on dissipative and m -dissipative operators. Treatment of the parabolic problem of Section 6.10 requires in addition results from Section 1.6. If one wants to include approximation results one can use the material of Chapter 10 or just the material of Section 10.1.

d) In order to present the results of Chapter 7 one only needs in addition the material of Section 1.2 (specifically Theorem 1.10 and Proposition 1.8) together with some basic definitions of Chapter 6 (for instance the definition of DS-approximations given in Definition 6.6).

e) For a presentation of the material of Chapter 8 one only needs in addition the results of Section 1.5 and the concept of DS-approximations.

f) The material on scalar conservation laws (Section 9.2) and on the Navier–Stokes equations (Section 9.3) could be used for advanced seminars and require also additional material not presented in this monograph.

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more frequently than just occasionally absent minded mathematicians and thus introducing a reasonable amount of normality into their – the mathematicians – life.

Raleigh and Graz, December 2001

Kazufumi Ito
Franz Kappel

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