Algebraic Invariants of Links

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Algebraic Invariants of Links

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Preface

This book is intended as an introduction to links and a reference for the invariants of abelian coverings of link exteriors, and to outline more recent work, particularly that related to free coverings, nilpotent quotients and concordance. Knot theory has been well served with a variety of texts at various levels, but essential features of the multicomponent case such as link homotopy, I -equivalence, the fact that links are not usually boundary links, longitudes, the role of the lower central series as a source of invariants and the homological complexity of the many-variable Laurent polynomial rings are all generally overlooked. Moreover it has become apparent that for the study of concordance and link homotopy it is more convenient to work with disc links; the distinction is imperceptible in the knot theoretic case.

Invariants of these types play an essential role in the study of such difficult and important problems as the concordance classification of classical knots and the questions of link concordance arising from the Casson-Freedman analysis of topological surgery problems, and particularly in the applications of knot theory to other areas of topology. For instance, the extension of the Disc Embedding Lemma to groups of subexponential growth by Freedman and Teichner derived from computations using link homotopy and the lower central series. Milnor's interpretation of the multivariable Alexander polynomial as a Reidemeister-Franz torsion was refined by Turaev, to give "sign-determined" torsions and Alexander polynomials. These were used by Lescop to extend the Casson invariant to all closed orientable 3-manifolds, and by Meng and Taubes to identify the Seiberg-Witten invariant for 3-manifolds. The multivariable Alexander polynomial

also arises in McMullen's lower bound for the Thurston norm of a torally bounded 3-manifold.

Links and the main equivalence relations relating them are defined in Chapter 1. (In particular, we include a proof of Giffen's theorem relating F -isotopy and I -equivalence via shift-spinning). In Chapter 2 we review homology and cohomology with local coefficients, and Poincaré duality for covering spaces. The most useful manifestations of duality are the Blanchfield pairings for abelian coverings (considered in this chapter) and for free coverings of homology boundary links (considered in Chapter 9). Most of Chapter 3 is on the determinantal invariants of modules over a commutative noetherian ring (including the Reidemeister-Franz torsion for chain complexes), but it also considers some special features of low-dimensional rings and Witt groups of hermitean pairings on torsion modules. These results are applied to the homology of abelian covers of link exteriors in the following five chapters. Chapter 4 is on the maximal abelian cover. Some results well-known for knots are extended to the many component case, and the connections between various properties of boundary links are examined. Relations with the invariants of sublinks, the total linking number cover, fibred links and finite abelian branched covers are considered in Chapter 5.

In the middle of the book (Chapters 6-8) the above ideas are applied in some special cases. Chapters 6 and 7 consider in more detail invariants of knots and of 2-component links, respectively. Here there are some simplifications, both in the algebra and the topology. In particular, surgery is used to describe the Blanchfield pairing of a classical knot (in Chapter 6) and to give Bailey's theorem on presentation matrices of the modules of 2-component links (in Chapter 7). Symmetries of links and link types, as reflected in the Alexander invariants, are studied in Chapter 8.

The later chapters (9-12) describe some invariants of nonabelian coverings and their application to questions of concordance and link homotopy. The links of greatest interest here are those concordant to sublinks of homology boundary links ($cSHB$ links). The exteriors

of homology boundary links have covers with nontrivial free covering group. As free groups have cohomological dimension 1, the ideas used in studying knot modules extend readily to the homology modules and duality pairings of such covers. This is done in Chapter 9, which may be considered as an introduction to the work of Sato, Du Val and Farber on high dimensional boundary links. We also give a new proof of Gutiérrez' unlinking theorem for n -links, which holds for all $n \geq 3$ and extends, modulo s -cobordism, to the case $n = 2$.

Although $cSHB$ links do not always have such free covers, their groups have nilpotent quotients isomorphic to those of a free group. More generally, the quotients of a link group by the terms of its lower central series are concordance invariants of the link. (The only other such invariants known are the Witt classes of duality pairings on covering spaces). Chapter 10 considers the connections between the nilpotent quotients, Lie algebra, cohomology algebra and minimal model of a group and more particularly the relations between Massey products and Milnor invariants for a link group. Although we establish the basic properties of the Milnor invariants here, we refer to Cochran's book for further details on geometric interpretations, computation and construction of examples.

The final two chapters are intended as an introduction to the work of Levine (on algebraic closure and completions), Le Dimet (on high dimensional disc links) and Habegger and Lin (on string links). As this work is still evolving, and the directions of further development may depend on the outcome of unproven conjectures, some arguments in these chapters are only sketched, if given at all. One of the difficulties in constructing invariants for links from the duality pairings of covering spaces is that, in contrast to the knot theoretic case, link groups do not in general share a common quotient with reasonable homological properties. The groups of all μ -component 1-links with all Milnor invariants 0 and the groups of all μ -component n -links for any $n \geq 2$ share the same tower of nilpotent quotients. The projective limit of this tower is the nilpotent completion of the free group on μ generators, and is uncountable. This is

related to other notions of completion in Chapter 11. Another problem is that the set of concordance classes of links does not have a natural group structure. However “stacking” with respect to the last coordinate endows the set of concordance classes of n -disc links with such a structure. Chapter 12 considers disc links and their relation to spherical links.

The emphasis is on establishing algebraic invariants and their properties, and constructions for realizing such invariants have been omitted, for the most part. The reader is assumed to know some algebraic and geometric topology, and some commutative algebra (to the level of a first graduate course in each). We occasionally use spectral sequence arguments. Commutative and homological algebra are used systematically, and we avoid as far as possible accidental features, such as the existence of Wirtinger presentations. While the primary focus is on links in S^3 , links in other homology spheres and higher dimensions and disc links in discs are also considered.

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Jonathan Hillman