PROBABILITY AND SCHRÖDINGER'S MECHANICS

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“But the chemists and the whole class of mechanics and empirics, should they have the temerity to attempt contemplation and philosophy, *being accustomed to meticulous subtlety in a few things*, they twist by extraordinary means all the rest into conformity with them and promote opinions more odious and unnatural than those advanced by the very rationalists.”

— Francis Bacon, *Preface to Natural History*, 1609 (my emphasis)

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Preface

The presentation and interpretation of (non-relativistic) quantum mechanics is a very well-worked area of study; there have to be very good reasons for adding to the literature on this subject.

My reasons are (obviously) that I am far from satisfied with much of the published work and find difficulties with some points, in particular:

- Any abstract formalism is much less rich than the structure from which it has been abstracted; a fact that even Wittgenstein had to come to terms with in the latter part his adult life.
Language is richer than (cannot be reduced to) a representation of logic, Schrödinger's mechanics is richer than (cannot be reduced to) a representation of Hilbert space. Just as language contains more structures than logic so Schrödinger's mechanics contains more structures than those of Hilbert space.
- The use of probability in quantum theory is arbitrary, eccentric, out of step with modern probability theory and is the source of the majority of "paradoxes" in the interpretation of quantum theory. While these paradoxes are the bread and butter of some of the more popular expositions of quantum theory, I cannot say that I am fond of paradoxes in physical theories.
- Although positivism is discredited as a philosophy of science it has left a huge clutter of verbal and conceptual debris strewn across the field of quantum theory. Positivism in its most aggressive form (instrumentalism) makes the mistake of confusing the meaning of a concept with the way in which numerical values of the variables involved in that concept may be determined. This attitude has, of course, added to the confusion about probabilities; "defining" them in terms of the ways in which they might be measured thus reinforcing the view that probabilities *are*

frequency ratios and the everyday opinion that probabilities are applicable to individual events.

- More mundanely, the prescriptions for generating Schrödinger's mechanics from classical mechanics in the vast majority of texts are wrong; they simply do not work. We are saved from total chaos by the fact that the form of the Schrödinger equation is known and used *independently* of these formal prescriptions by working scientists. I am at a loss to explain why this central point is ignored in text after text, both on the applications of quantum theory and on its interpretation.

Most modern studies of the interpretation of quantum mechanics are "modern" in another sense; that used in literary criticism. They study the *theory* of quantum mechanics rather than the *quantum mechanics* of the energetics and distribution of sub-atomic particles. Many of these works concentrate on the alleged consequences of imaginary experiments involving spin angular momentum; uncritically using the colloquial, "everyday" interpretation of probability rather than the modern mathematical theory. Indeed, scarcely any text says what is meant by "probability", leaving the reader to assume that the everyday interpretation is correct. If a Hamiltonian is used at all in these expositions it is an empirical one using "coupling constants" and making no reference to the laws of interaction in physical systems.

In contrast, I wish to present a study of what I have called Schrödinger's mechanics, the richest, most concrete and most thoroughly *interpretable* of a variety of more-or-less abstract structures falling under the umbrella of "quantum mechanics". Quantum theory is a mathematically articulated theory embedded in a historical stream of scientific thought; it cannot be understood or its interpretation appreciated without addressing these obvious facts. I shall not shrink from joining scientists in cosmology, geology, biology, chemistry, history, archeology, sociology, paleontology, (even physics) in using, where possible, a *mechanistic* mode of explanation occasionally involving unobserved, even unobservable, objects and processes. What is more, I will occasionally look at actual solutions of the (spatial differential) equations to examine how they are capable of interpretation and how this interpretation bears on the referent of Schrödinger's mechanics.

In a previous work I chose¹ to express myself using the "ensemble" interpretation of probabilistic statements since this interpretation has been

---

¹With considerable reservations, as I explained in an Appendix to the work.

one of the major strands in quantum theory (notably by Einstein) and because it is closest to the “abstract object” interpretation which I use in this work. My main aim was twofold: to stay in contact with one of the main streams of thought in quantum theory and, more important, to emphasise the incorrectness of the colloquial interpretation of probability which assumes that probability statements apply to single concrete systems. I now regret this, since even the ensemble interpretation, although vastly superior to the colloquial interpretation, cannot be made precise enough for my purposes here.

Two of the main lessons of the past century in the investigation of the structure of mathematics and, in general, of mathematically articulated theories have been the failures of ordinary logic and intuition for:

- Systems with an infinite number of members or for infinite combinations of statements,
- Constructions which are self-referential like the ones which generate Russell’s paradox or for systems whose logic may be “internalised” like Gödel’s theorem.

So, generally speaking, I shall not be concerned with “the wave function of the Universe”, “the wave function of the measuring apparatus” or even “the wave function of the observer” since the first is both recursively self-referential and with infinite referent and the others arguably so. I shall, therefore, try to avoid writing down symbols for non-existent mathematical objects; the typographical equivalent of hot air. The science of the very small and very light is quite complex enough for me, I do not have time (literally or figuratively), or indeed the expertise, to express opinions about the bearing of quantum theory on God, Mind or Consciousness.

My aim is entirely bourgeois in the sense used by Wittgenstein and by Marx. I try to solve the problems of Schrödinger’s mechanics “from within”; I do not wish to sweep away quantum theory but to bring this most successful physical theory of the twentieth century back into the mainstream of the great tradition of the mechanics of Newton, Lagrange and Hamilton and to base its interpretation on the work of a pillar of twentieth century mathematics: A. N. Kolmogorov.

The degeneration of the interpretation of modern science into silopsism and triviality requires an entirely different kind of investigation for which I have neither the qualifications nor the stomach.

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Organisation

As this work is presented in a rather unusual way, this short chapter gives both a description of the organisation of material and an attempted justification for this choice.

Analysis and interpretation of Schrödinger's mechanics is a very "mature" subject in the sense that an enormous amount has been written about the matter without any consensus emerging. Or, rather, such consensus as there is about the interpretation of quantum theory is, in my view, based on an erroneous (pre-Kolmogorov) view of probability. So, in what is written here almost every page (indeed, in places, almost every sentence) will contain material that is in dispute. Using Schrödinger's starting point and Kolmogorov's probability theory, I shall say (and I shall prove), among other things, that:

- Probability statements do not refer to individual objects
- Probability distributions do not "collapse"
- The Hamiltonian operator cannot be obtained by an operator substitution in the classical Hamiltonian function
- Dynamical variables may or may not all be represented by linear operators but there is no general way to find the form of these operators
- Conserved quantities are not always represented by operators which mutually commute
- It is nonsensical to give a physical interpretation to the terms in a perturbation expansion

To take the time and space to discuss the opposing views on every such point is tiresome and, more importantly, makes the presentation diffuse and lacking in direction. I shall therefore simply give my presentation

of probability and Schrödinger's mechanics without detailed references to other points of view or, indeed, in most cases even giving an opposing view a "fair hearing".

This device will enable the presentation of the theory and interpretation of Schrödinger's mechanics "as if" one were developing an interpreted theory *ab initio* in the cleanest and most logical manner while conveniently ignoring other viewpoints. My defence for this outrageous choice is twofold: defending my position would simply take too much space and the alternative and opposing views are too well-known to need rehearsing here.

Throughout the work I use the term "state function" rather than the familiar "wave function"; this is a deliberate attempt to avoid, even in terminology, the celebrated wave/particle duality. In fact the solutions of the Schrödinger equation are only waves in the extremely atypical case of a particle in the absence of a field of force (a "free" particle). And even these waves are not real physical waves but particular forms of a function which generates a probability distribution. I have also tended (as I say on page 148 of Chapter 8) to work in a system of units — "atomic units" — in which Planck's constant has the value unity unless the historical context explicitly demands the appearance of h .