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Preface

Ordinary Differential Equations (ODE) is a well-developed discipline. It is a classical subject for most graduate students to study if they want to learn analysis and applied mathematics. There are many standard texts of ODE, for instance, the books, Coddington and Levinson [CL], Hartman [Ha], J. Hale [H], Miller and Michel [MM]. Based on my past twenty years of teaching experience of graduate course in ODE, I find that for most graduate students these books are either too difficult to understand or too abstract to follow. This is the reason why I have decided to write my own ODE lecture notes. This book is aimed to present the classical ODE materials in an easy way with rigorous proofs. In this text many examples from mathematical biology and physical science to interpret the meanings and the applications of the theorems have been given. Besides the classical materials, Brouwer degree in finite dimensional space following the index theory has been discussed in Chapter 8. Brouwer degree is a powerful topological tool for studying the nonlinear analysis. Introducing Brouwer degree paves the way to the Leray-Schauder degree in infinite dimensional space. In Chapter 9 we introduce perturbation method, the regular perturbation and singular perturbation methods which are very important in applied mathematics.

This book is designed for a one-year graduate course. Let's briefly state the contents of each chapter in this book.

Chapter 1: Introduction. In this chapter important examples of nonlinear systems of ODEs from physical science and mathematical biology have been introduced.

Chapter 2: Fundamental Theory. In this chapter I first prove the local existence and uniqueness of solutions of initial value problem of ODE. Then the continuation of the solutions and the global existence of the solutions are studied. The continuous dependence on initial conditions and parameters is proved. Applying continuous dependence on initial conditions and parameters gives the results of scalar differential inequalities. The differen-

tial inequalities for systems are also discussed.

Chapter 3: Linear Systems. In this chapter we discuss the fundamental matrices of general linear systems $x' = A(t)x$. Then for the linear system with constant coefficients $x' = Ax$ we introduce exponential matrix e^{At} as a fundamental matrix. For periodic linear system $x' = A(t)x$, $A(t + \omega) = A(t)$, $\omega > 0$, we study the structure of the solutions by proving Floque's Theorem. For two-dimensional linear autonomous $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, we introduce the important notions of node, focus, center and saddle. Similar to linear algebra, we introduce adjoint systems of $x' = Ax$. We obtain important results similar to the fundamental theorem of linear algebra.

Chapter 4: Stability of Nonlinear Systems. In this chapter we introduce concepts of stability, asymptotic stability and instability of an equilibrium solution. To verify these stabilities, we introduce the method of linearization by checking the eigenvalues of variational matrix. When the equilibrium is a saddle, we prove the existence of stable and unstable manifolds. For periodic orbit, we introduce the concept of orbit stability which is different from the asymptotic stability. The results related to Floquet multipliers have been discussed in Chapter 3. In section 4.5 we study the existence of travelling wave solutions of the well-known partial differential equations, the Fisher's equation and the bistable equation by applying the Stable Manifold Theorem.

Chapter 5: Method of Lyapunov Functions. In this chapter we first introduce the concept of dynamical system, α -limit set, ω -limit set and their properties in abstract setting. In fact, autonomous ODE system is an important example of dynamical system. Then we discuss Lyapunov functions and their use in determining the global asymptotic stability or the domain of attraction of a locally stable equilibrium. We prove the LaSalle's invariance principle which explains how to apply Lyapunov function to locate ω -limit set.

Chapter 6: Two-Dimensional Systems. In this chapter we prove one of the most important theorems in nonlinear dynamics, Poincaré-Bendixson Theorem which states that a bounded trajectory of a two-dimensional system either converges to an equilibrium or approaches a limit cycle. We also discuss Dulac Criterion which provides a method to eliminate the existence of periodic solutions. In section 6.2 we discuss the Levinson-Smith Theorem which proves the uniqueness of limit cycle for Lienard equation. An important example of Lienard equation is van der Pol equation. We discuss the relaxation property of the van der Pol oscillators. In section 6.3, we introduce Hopf Bifurcation Theorem which is an important tool to detect

the existence of periodic solution for autonomous system $x' = f(\lambda, x)$ where λ is a bifurcation parameter.

Chapter 7: Second Order Linear Equations. In this chapter we first discuss the Sturm's Comparison Theorem which is an important tool to study oscillation properties of a solution of a second order linear equation. Similar to the eigenvalue problems $Ax = \lambda x$, we consider the Sturm-Liouville boundary value problem and prove the existence of infinitely many discrete eigenvalues and show that the corresponding eigenfunctions satisfy the "node" properties. For second order nonlinear equation $x'' = f(t, x, x')$, $x(a) = x_0$, $x(b) = x_1$, we introduce Green's function so that we convert the boundary value problem into an integral equation.

Chapter 8: The Index Theory and Brouwer Degree. In this chapter we first introduce the Index $I_f(C)$ for a vector field f in the plane and a simple closed curve C . Then we prove various properties of index and apply index theory to prove fundamental theorem of algebra and Brouwer fixed point theorem. In section 8.2 we generalize the index $I_f(C)$ to the topological degree in finite dimensional space, the Brouwer degree in R^n . Topological degree is an important tool in studying nonlinear problems.

Chapter 9: Introduction to Regular and Singular Perturbation Methods. Perturbation methods are very important in applied mathematics. In this chapter we explain why regular perturbation method works by using Implicit Function Theorem and Fredholm Alternatives. Then we discuss singular perturbation methods for both boundary value problems and initial value problems.

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