NON-AUTONOMOUS KATO CLASSES AND FEYNMAN–KAC PROPAGATORS

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Archil Gulisashvili
Ohio University, USA

Jan A van Casteren
University of Antwerp, Belgium

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To my wife, Olga Molchanova, and my two sons,
Oleś and Misha

Archil Gulisashvili

To my wife, Riet Wesselink, my daughter Linda,
my son Wilfred, and my granddaughter Elly

Jan A. van Casteren

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Preface

This book covers selected topics from propagator theory. Propagators and backward propagators are two-parameter families of linear operators satisfying special conditions. Propagators satisfy the “flow” conditions, while backward propagators satisfy the “backward flow” conditions. Examples of propagators abound in mathematical physics, partial differential equations, and probability theory. For instance, solutions to Cauchy problems for non-autonomous evolution equations are generated by propagators. Important special examples are initial value problems for the heat equation perturbed by low order terms with time-dependent coefficients. A rich source of backward propagators is probability theory, where backward propagators arise as families of integral operators associated with transition probability functions. Such backward propagators are called free backward propagators. They admit a probabilistic characterization in terms of non-homogeneous Markov processes associated with transition probability functions.

Although propagators have many similarities with semigroups, propagator theory is not yet as complete as semigroup theory. One of the main obstacles in our understanding of propagators is their non-commutativity. Even the term “propagator” is not standard, and several other names have appeared in the mathematical literature, e.g., evolution families, solution operators, non-autonomous semigroups, etc. The present book is mainly devoted to free propagators, Feynman-Kac propagators, and related topics such as non-homogeneous Markov processes, reciprocal processes, and non-autonomous Kato classes of functions and measures. Since the selection of topics covered in the book was influenced by the research interests of the authors, many important subjects have been omitted. We refer the reader to Notes and Comments sections at the end of every chapter for additional information and the lists of references. In our opinion, the book is accessi-

ble to advanced graduate students interested in semigroup theory, partial differential equations, and probability theory. It is also suited for researches in these areas.

Chapter 1 contains basic facts about non-homogeneous Markov processes. The chapter includes stochastic processes with values in separable locally compact spaces, measurable and progressively measurable processes, one-sided continuity and continuity of sample paths, and space-time processes. We also discuss time-reversal, reciprocal processes, and Brownian and Cauchy bridges.

Chapter 2 introduces propagators and backward propagators. It begins with a discussion of the continuity properties of propagators. Topics included in this chapter are right and left generators of propagators, Kolmogorov's forward and backward equations, Howland semigroups associated with propagators, free backward propagators generated by transition probability functions, the strong Markov property of non-homogeneous stochastic processes, and Feller type properties of propagators. Several sections are devoted to standard processes associated with backward Feller-Dynkin propagators.

Chapter 3 studies non-autonomous Kato classes of functions and measures. These classes are generalizations of the Kato class of potential functions introduced by M. Aizenman and B. Simon. After discussing additive functionals of stochastic processes associated with time-dependent measures we give a probabilistic description of the Kato classes using these functionals. Various exponential estimates for non-autonomous additive functionals are obtained in the chapter. We also discuss a known fact that under certain restrictions fundamental solutions to second order conservative parabolic partial differential equations are transition probability densities. The chapter contains basic information on stochastic integration, stochastic differential equations, and diffusion processes.

Chapter 4 studies Feynman-Kac propagators. It starts with a brief review of Schrödinger semigroups with Kato class potentials. Topics covered in the chapter include Feynman-Kac propagators and Howland semigroups associated with them, the integral kernels of Feynman-Kac propagators, and Duhamel's formula. We also discuss the following problem: Determine what properties of free propagators are inherited by their Feynman-Kac perturbations. Among the properties appearing in the inheritance problem are the L^r -boundedness, the $(L^r - L^q)$ -smoothing property, the Feller property, the Feller-Dynkin property, and the BUC-property. We end the book by proving that under certain restrictions the Feynman-Kac propagators

generate viscosity solutions to non-autonomous Cauchy problems.

Most of the book was written during our visits to the University of Antwerp, Ohio University, and Centre de Recerca Matemàtica (CRM) in Bellaterra (Barcelona), Spain. We would like to thank the faculty and staff of the Department of Mathematics and Computer Science at the University of Antwerp, the Department of Mathematics at Ohio University, and CRM for their wonderful hospitality. We acknowledge the financial support of FWO Flanders (FWO Research Network WO.011.96N). The first-named author's research at CRM was supported by the grant "Beca de profesores e investigadores extranjeros en régimen de año sabático del Ministerio de Educación, Cultura y Deporte de España, referencia SAB2002-0066", and the second-named author acknowledges the financial support of the Department of Mathematics at Ohio University during one of his visits to Ohio University. We are very grateful to these universities and research institutions for their support.

A. Gulisashvili and J. A. van Casteren

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