Lectures on Algebra

Volume I

This page is intentionally left blank

S S Abhyankar

Purdue University, USA

Lectures on Algebra

Volume I

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

LECTURES ON ALGEBRA

Volume I

Copyright © 2006 by World Scientific Publishing Co. Pte. Ltd.

All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the Publisher.

For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher.

ISBN 981-256-826-3

PREFACE

In universities and colleges it has become customary to give two algebra courses, the first being called abstract algebra and the second linear algebra. The present volume, *Lectures on Algebra I*, is meant as a text-book for an abstract algebra course, while the forthcoming sequel, *Lectures on Algebra II*, should serve as a text-book for a linear algebra course. The author's fondness for algebraic geometry shows up in both volumes, and his recent preoccupation with the application of group theory to the calculation of Galois groups is evident in the second volume which contains more local rings and more algebraic geometry. Both volumes are based on the author's lectures at Purdue University during the last several years. An attempt has been made to make these volumes self-contained. The reader may prefer to start with the sixth lecture which gives a rapid summary of the first five lectures. He may also find it helpful to look at the detailed contents printed at the end of the volume just before the index; this is particularly significant for the enormous (about 300 pages) Section §5 of Lecture L5.

When in a certain lecture we are referring to an item from another lecture, the citation of the other lecture precedes the citation of the item. Thus, for instance, in the proof of Theorem (Q4)(T13) in Lecture L5, the reference L4§5(O11) is to Observation (O11) of §5 of Lecture L4, whereas the reference (T11) is to Theorem (T11) of Lecture L5.

Frequently, assertions made in one place are proved or expanded later on. For this purpose, forward reference is indicated by [cf.]. For instance, on page 4, at the end of the sentence "This is unique up to isomorphism, i.e., between two copies of it there is a one-to-one onto map preserving sums and products;" the phrase "[cf. L5§5(Q32)(T138.2)]" means that the proof of the preceding claim that there is a unique field $\text{GF}(q)$ of q elements, will be given in Theorem (T138.2) of Quest (Q32) of Section §5 of Lecture L5.

Like a Russian Petrushka doll, there are many books within this book. For instance, the first three lectures, L1, L2, L3 constitute a booklet on a basic abstract algebra course. The sixth lecture L6 by itself constitutes another such course. These two alternatives together make up a larger such booklet. The fourth lecture L4 is a booklet on commutative algebra. It is continued in the fifth lecture L5 which may be

viewed as a treatise on commutative algebra. Within it, Quest (Q31) is a pamphlet on Suslin's work on projective modules and special linear groups over multivariable polynomial rings. Finally, Sections §§2-5 of L3, §§8-9 of L4, and Quests (Q33)-(Q35) of L5§5 form a short course on algebraic geometry.

Thanks are due to Sudhir Ghorpade, Nan Gu, Nick Inglis, Valeria Grant Perez, Avinash Sathaye, David Shannon, Balwant Singh, Umud Yalcin, and Ikkwon Yie for much help. Thanks are also due to NSF Grant DMS 99-88166 and NSA Grant MSP H98230-05-1-0040 for financial support.

Shreeram S. Abhyankar,
Mathematics Department,
Purdue University,
West Lafayette, IN 47907, USA,
e-mail: ram@cs.purdue.edu

CONTENTS

Lecture L1: QUADRATIC EQUATIONS	1
§1: Word Problems	1
§2: Sets and Maps	3
§3: Groups and Fields	3
§4: Rings and Ideals	6
§5: Modules and Vector Spaces	8
§6: Polynomials and Rational Functions	9
§7: Euclidean Domains and Principal Ideal Domains	13
§8: Root Fields and Splitting Fields	14
§9: Advice to the Reader	16
§10: Definitions and Remarks	16
§11: Examples and Exercises	23
§12: Notes	27
§13: Concluding Note	29
Lecture L2: CURVES AND SURFACES	30
§1: Multivariable Word Problems	30
§2: Power Series and Meromorphic Series	34
§3: Valuations	39
§4: Advice to the Reader	43
§5: Zorn's Lemma and Well Ordering	44
§6: Utilitarian Summary	52
§7: Definitions and Exercises	52
§8: Notes	59
§9: Concluding Note	60
Lecture L3: TANGENTS AND POLARS	61
§1: Simple Groups	61
§2: Quadrics	63
§3: Hypersurfaces	64
§4: Homogeneous Coordinates	66

§5: Singularities	70
§6: Hensel's Lemma and Newton's Theorem	72
§7: Integral Dependence	77
§8: Unique Factorization Domains	81
§9: Remarks	82
§10: Advice to the Reader	83
§11: Hensel and Weierstrass	83
§12: Definitions and Exercises	90
§13: Notes	98
§14: Concluding Note	98
 Lecture L4: VARIETIES AND MODELS	 100
§1: Resultants and Discriminants	100
§2: Varieties	104
§3: Noetherian Rings	105
§4: Advice to the Reader	107
§5: Ideals and Modules	108
§6: Primary Decomposition	134
§6.1: Primary Decomposition for Modules	136
§7: Localization	137
§7.1: Localization at a Prime Ideal	144
§8: Affine Varieties	146
§8.1: Spectral Affine Space	152
§8.2: Modelic Spec and Modelic Affine Space	152
§8.3: Simple Points and Regular Local Rings	153
§9: Models	154
§9.1: Modelic Proj and Modelic Projective Space	157
§9.2: Modelic Blowup	159
§9.3: Blowup of Singularities	160
§10: Examples and Exercises	161
§11: Problems	171
§12: Remarks	172
§13: Definitions and Exercises	195
§14: Notes	200
§15: Concluding Note	201
 Lecture L5: PROJECTIVE VARIETIES	 202
§1: Direct Sums of Modules	202
§2: Grades Rings and Homogeneous Ideals	206
§3: Ideal Theory in Graded Rings	209
§4: Advice to the Reader	216
§5: More about Ideals and Modules	216

(Q1) Nilpotents and Zerodivisors in Noetherian Rings	216
(Q2) Faithful Modules and Noetherian Conditions	218
(Q3) Jacobson Radical, Zariski Ring, and Nakayama Lemma	219
(Q4) Krull Intersection Theorem and Artin-Rees Lemma	220
(Q5) Nagata's Principle of Idealization	225
(Q6) Cohen's and Eakin's Noetherian Theorems	229
(Q7) Principal Ideal Theorems	230
(Q8) Relative Independence and Analytic Independence	236
(Q9) Going Up and Going Down Theorems	241
(Q10) Normalization Theorem and Regular Polynomials	247
(Q11) Nilradical, Jacobson Spectrum, and Jacobson Ring	261
(Q12) Catenarian Rings and Dimension Formula	268
(Q13) Associated Graded Rings and Leading Ideals	272
(Q14) Completely Normal Domains	277
(Q15) Regular Sequences and Cohen-Macaulay Rings	280
(Q16) Complete Intersections and Gorenstein Rings	300
(Q17) Projective Resolutions of Finite Modules	311
(Q18) Direct Sums of Algebras, Reduced Rings, and PIRs	340
(Q18.1) Orthogonal Idempotents and Ideals in a Direct Sum	341
(Q18.2) Localizations of Direct Sums	344
(Q18.3) Comaximal Ideals and Ideal Theoretic Direct Sums	345
(Q18.4) SPIRs = Special Principal Ideal Rings	348
(Q19) Invertible Ideals, Conditions for Normality, and DVRs	354
(Q20) Dedekind Domains and Chinese Remainder Theorem	364
(Q21) Real Ranks of Valuations and Segment Completions	372
(Q22) Specializations and Compositions of Valuations	381
(Q23) UFD Property of Regular Local Domains	385
(Q24) Graded Modules and Hilbert Polynomials	393
(Q25) Hilbert Polynomial of a Hypersurfaces	397
(Q26) Homogeneous Submodules of Graded Modules	399
(Q27) Homogeneous Normalization	401
(Q28) Alternating Sum of Lengths	408
(Q29) Linear Disjointness and Intersection of Varieties	414
(Q30) Syzygies and Homogeneous Resolutions	433
(Q31) Projective Modules Over Polynomial Rings	441
(Q32) Separable Extensions and Primitive Elements	514
(Q33) Restricted Domains and Projective Normalization	529
(Q34) Basic Projective Algebraic Geometry	534
(Q34.1) Projective Spectrum	534
(Q34.2) Homogeneous Localization	536
(Q34.3) Varieties in Projective Space	541
(Q34.4) Projective Decomposition of Ideals and Varieties	545

(Q34.5) Modelic and Spectral Projective Spaces	547
(Q34.6) Relation between Affine and Projective Varieties	548
(Q35) Simplifying Singularities by Blowups	552
(Q35.1) Hypersurface Singularities	552
(Q35.2) Blowing-up Primary Ideals	553
(Q35.3) Residual Properties and Coefficient Sets	555
(Q35.4) Geometrically Blowing-up Simple Centers	555
(Q35.5) Algebraically Blowing-up Simple Centers	559
(Q35.6) Dominating Modelic Blowup	566
(Q35.7) Normal Crossings, Equimultiple Locus, and Simple Points	567
(Q35.8) Quadratic and Monoidal Transformations	569
(Q35.9) Regular Local Rings	577
§6: Definitions and Exercises	578
§7: Notes	596
§8: Concluding Note	597
 Lecture L6: PAUSE AND REFRESH	 598
§1: Summary of Lecture L1 on Quadratic Equations	598
§2: Summary of Lecture L2 on Curves and Surfaces	603
§3: Summary of Lecture L3 on Tangents and Polars	606
§4: Summary of Lecture L4 on Varieties and Models	608
§5: Summary of Lecture L5 on Projective Varieties	611
§6: Definitions and Exercises	634
 BIBLIOGRAPHY	 689
 DETAILED CONTENT	 691
 NOTATION-SYMBOLS	 713
 NOTATION-WORDS	 717
 INDEX	 725