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INTRODUCTION TO GENERAL RELATIVITY

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Preface

Eleven years after Einstein's theory of special relativity completely changed our understanding of the relationship between space and time [Einstein (1905)], his theory of general relativity revolutionized our understanding of how mass and energy change the underlying space-time structure of the physical universe [Einstein (1916)].

Some time ago I had to learn general relativity in connection with my research. I had great difficulty listening to the experts and understanding what they were doing. I decided I had to teach the subject to myself, and I proceeded to do so. Much later, I converted my notes to a semester course at William and Mary, which was offered three times. It was aimed at physics graduate students, but many undergraduates participated, and excelled. The students seemed to learn from it and enjoy it, and the outcome was very satisfying. I decided that I would convert these lectures into a book entitled *Introduction to General Relativity*.

General relativity is a difficult subject for two reasons. The first is that the math is unfamiliar to most physics students, and the second is that since the four-dimensional coordinate system has no intrinsic meaning, it is difficult to get at the physical interpretation and physical consequences of any result. The goals of the present text are as follows:

- The book is aimed at physics graduate students and advanced undergraduates. Only a working knowledge of classical lagrangian mechanics is assumed,¹ although an acquaintance with special relativity will make the book more meaningful. Within this framework, the material is self-contained;
- The necessary mathematics is developed within the context of a

¹As presented, for example, in [Fetter and Walecka (2003)].

concrete physical problem — that of a point mass constrained to move without friction on an arbitrarily shaped two-dimensional surface;

- A strong emphasis is placed on the physical interpretation and physical consequences in all applications;
- The book is not meant to be a weighty tome for experts (indeed, I could not write one), but, in my opinion, most of the interesting applications of GR are covered;
- The final “Special Topics” chapter takes the reader up to a few areas of active research.

At the end of the course at William and Mary, students selected a topic of current interest, wrote a term paper, and then gave a talk to the class in the format of a contributed session of an American Physical Society meeting (complete with abstract). I was amazed and pleased with the level at which the students were able to perform. It is my sincere hope that this text will provide useful background for other young people and aid them in their exploration of many of the fascinating modern developments based on Einstein’s theory of general relativity.

An extensive bibliography has not been attempted, although most of the comparable and more advanced texts are referenced, as are relevant websites. Since general relativity is an old subject, no effort has been made to trace the origin of the examples and problems, many of which are modified versions of those in the list of texts and monographs.

I was delighted when World Scientific Publishing Company, which had done an exceptional job with two of my previous books, showed enthusiasm for publishing this new one. I would like to thank Dr. K. K. Phua, Executive Chairman of World Scientific Publishing Company, and my editor Ms. Lakshmi Narayanan, for their help and support on this project.

I would also like to thank my colleagues Paolo Amore, Brian Serot, and J. Wallace Van Orden for their reading of the manuscript.

Williamsburg, Virginia
January 3, 2007

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Contents

<i>Preface</i>	vii
1. Introduction	1
2. Particle on a Two-Dimensional Surface	9
2.1 Infinitesimal Displacements	10
2.2 Lagrange's Equations	12
2.3 Reciprocal Basis	14
2.4 Geodesic Motion	19
2.5 Role of Coordinates	24
3. Curvilinear Coordinate Systems	27
3.1 Line Element	27
3.2 Reciprocal Basis	29
3.3 Metric	29
3.4 Vectors	30
3.5 Tensors	31
3.6 Change of Basis	33
3.7 Transformation Law	36
3.8 Affine Connection	39
3.9 Example: Polar Coordinates	41
4. Particle on a Two-Dimensional Surface—Revisited	43
4.1 Motion in Three-Dimensional Euclidian Space	43
4.2 Parallel Displacement	46

5.	Some Tensor Analysis	51
5.1	Covariant Derivative	51
5.2	The Riemann Curvature Tensor	54
5.3	Second Covariant Derivative	60
5.4	Covariant Differentiation	64
5.5	Symmetry Properties of the Riemann Tensor	67
5.6	Bianchi Identities	68
5.7	The Einstein Tensor	69
5.8	Example—Surface of a Sphere	72
5.9	Volume element	76
6.	Special Relativity	87
6.1	Basic Principles	87
6.2	Four-Vectors	88
6.3	Relativistic Particle Motion	91
6.4	Lorentz Transformations	98
6.5	General Tensor Transformation Law	109
6.6	Relativistic Hydrodynamics	111
6.7	Transition to General Relativity	120
7.	General Relativity	123
7.1	Einstein’s Theory	123
7.2	Newtonian Limit	126
7.3	The Equivalence Principle	131
7.4	Local Freely Falling Frame (LF^3)	133
7.5	Spherically Symmetric Solution to Field Equations	135
7.6	Solution in Vacuum	143
7.7	Interpretation of Schwarzschild Metric	148
7.8	A Few Applications	152
8.	Precession of Perihelion	155
8.1	Lagrangian	156
8.2	Equations of Motion	159
8.3	Equilibrium Circular Orbits	163
8.4	Small Oscillations About Circular Orbits	164
8.5	Some Numbers and Comparison with Experiment	169
8.6	Deflection of Light	171

9. Gravitational Redshift	173
9.1 Basic Observation	174
9.2 Frequency Shift	174
9.3 Propagation of Light	176
9.4 A Cyclic Process	178
10. Neutron Stars	181
10.1 Hydrodynamics in General Relativity	181
10.2 Tolman-Oppenheimer-Volkoff (TOV) Equations	190
10.3 Relativistic Mean Field Theory of Nuclear Matter (RMFT)	201
10.4 Neutron Stars and Black Holes	210
11. Cosmology	217
11.1 Uniform Mass Background	218
11.2 Robertson-Walker Metric with $k = 0$	220
11.3 Solution to Einstein's Equations	225
11.4 Interpretation	229
11.5 Cosmological Redshift	233
11.6 Horizon	237
11.7 Some Comments	239
12. Gravitational Radiation	241
12.1 Linearized Theory	241
12.2 Auxiliary (Lorentz) Condition	245
12.3 Plane-Wave Solution to the Einstein Field Equations	250
12.4 Interpretation	253
12.5 Detection	257
13. Special Topics	263
13.1 Einstein-Hilbert Lagrangian	264
13.2 Cosmological Constant	267
13.3 Additional Scalar Field	268
13.4 Robertson-Walker Metric with $k \neq 0$	270
13.5 Inflation	275
14. Problems	281
Appendix A Reduction of $g^{\mu\nu} \delta R_{\mu\nu}$ to covariant divergences	321

Appendix B	Robertson-Walker Metric with $k \neq 0$	325
	<i>Bibliography</i>	327
	<i>Index</i>	331