Synchronization in Complex Networks *of* Nonlinear Dynamical Systems

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To Ann and Brian

For keeping my chaotic life in synch

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Preface

This book studies complete synchronization of coupled nonlinear systems in arbitrary networks and continues where the book “Synchronization in coupled chaotic circuits and systems” (World Scientific, 2002) left off. In particular, we delve more deeply into the connection between coupling topology and synchronization and focus on the graph-theoretical properties of the underlying topology. Another aspect of this book is to study how properties of complex network models such as small world models or preferential attachment models affect the synchronization properties of a network of dynamical systems coupled via such graphs. This area of research is experiencing tremendous growth of activity and no doubt many new results will have appeared by the time you read this and we apologize beforehand for this book’s incompleteness.

This book would not have been possible without the discussions, advice and encouragement of many friends and colleagues. I would especially like to thank Ming Cao, Guanrong Chen, Leon Chua, Don Coppersmith, Alan Hoffman, Ying Huang, Ljupco Kocarev, Tamás Roska, Mike Shub, and Charles Tresser for stimulating intellectual interactions over the years.

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