Supermanifolds

Theory and Applications

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Alice Rogers

King's College London

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to Richard

Preface

The subject of supermanifolds has to some extent ‘just grewed’ like Topsy, as the idea of adjoining anticommuting variables to conventional commuting variables proved useful in a variety of contexts. This has led to a bewildering variety of approaches which has often obscured the underlying unity of the ideas. Supermanifolds have been with me through much of this period of growth, and now in writing this book I intend not a formal mathematical treatise, but rather a working man or woman’s guide to the geometry and analysis of supermanifolds, together with applications of the theory. Supermanifolds continue to find new uses, the underlying ideas have proved robust, powerful and adaptable. My aim is to provide a unified picture, distilling the key ideas from a welter of sources. I have tried to give references to relevant original work, and can only apologise for any failures, which will have been unintentional.

I am very grateful for many conversations about supermanifolds that I have enjoyed and benefited from over the years – these have been with almost everyone whose work is referred to in this book, a sad exception being Berezin. I would particularly like to thank Bryce DeWitt and Marjorie Batchelor, both of whom have patiently explained many things to me. I am also grateful to my colleagues in the mathematics department of King’s College London for a stimulating and agreeable working environment.

F. A. Rogers
King’s College London
2006

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