Stochastic Systems In Merging Phase Space

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Preface

“... the theory of systems should be built on the methods of simplification and is, essentially, the science of simplification”.
Walter Ashby (1969)

The actual problem of systems theory is the development of mathematically justified methods of simplification of complicated systems whose mathematical analysis is difficult to perform even with help of modern computers. The main difficulties are caused by the complexity of the phase (state) space of the system, which leads to virtually boundedless mathematical models.

A simplified model for a system must satisfy the following conditions: (i) The local characteristics of the simplified model are determined by rather simple functions of the local characteristics of original model. (ii) The global characteristics describing the behavior of the stochastic system can be effectively calculated on large enough time intervals. (iii) The simplified model has an effective mathematical analysis, and the global characteristics of the simplified model are close enough to the corresponding characteristics of the original model for application.

Stochastic systems considered in the present book are *evolutionary systems* in *random medium*, that is, *dynamical systems* whose state space is subject to random variations. From a mathematical point of view, such systems are naturally described by operator-valued stochastic processes on Banach spaces and are nowadays known as *Random evolution*.

This book gives recent results on stochastic approximation of systems by weak convergence techniques. General and particular schemes of proofs for average, diffusion, diffusion with equilibrium, and Poisson approximations of stochastic systems are presented. The particular systems studied here

are stochastic additive functionals, dynamical systems, stochastic integral functionals, increment processes, impulsive processes. As application we give the cases of absorption times, stationary phase merging, semi-Markov random walk, Lévy approximation, etc.

The main mathematical object of this book is a family of coupled stochastic processes $\xi^\varepsilon(t), x^\varepsilon(t), t \geq 0, \varepsilon > 0$ (where ε is the small series parameter) called the *switched* and *switching* processes. The switched process $\xi^\varepsilon(t), t \geq 0$, describes the system evolution, and, in general, is a stochastic functional of a third process. The switching process $x^\varepsilon(t), t \geq 0$, also called the *driving* or *modulation process*, is the perturbing process or the random medium, and can represent the environment, the technical structure, or any perturbation factor.

The two modes of switching considered here are Markovian and semi-Markovian. Of course, we could present only the semi-Markov case since the Markov is a special case. But we present both mainly for two reasons: the first is that proofs are simpler for the Markov case, and the second that most of the readers are mainly interested by the Markov case.

The switching processes are considered in *phase split and merging scheme*. The phase merging scheme is based on the split of the phase space into disjoint classes

$$E = \cup_{k \in V} E_k, \quad E_k \cap E_{k'} = \emptyset, \quad k \neq k' \quad (0.1)$$

and further merging these classes $E_k, k \in V$, into distinct states $k \in V$. So the merged phase space of the simplified model of system is $\hat{E} = V$ (see Figure 4.1).¹

The transitions (connections) between the states of the original system S are merged to yield the transitions between merged states of the merged system $\hat{S}$. The analysis of the merged system is thus significantly simplified.

It is important to note that the additional supporting system S^0 with the same phase space E but without connections between classes of states E_k is used.

Split the phase space (0.1) just means introducing a new supporting system consisting of isolated subsystems $S_k, k \in V$, defined on classes of states $E_k, k \in V$. The merged system $\hat{S}$ is constructed with respect to the ergodic property of the support system S^0 .

It is worth noticing that the initial processes in the series scheme contain no diffusion part. Diffusion processes appear only as limit processes.

¹Figures, theorems, lemmas, etc. are numbered by x.y, where x is the number of chapter, and y is the number of figure, theorem, etc. into the chapter.

The general scheme of proof of weak convergence for stochastic processes in series scheme is the following.

I. *Limit compensating operator:*

1. Construction of the compensating operator $\mathbb{L}^\varepsilon$ of the Markov additive process $\xi^\varepsilon(t), t \geq 0$.
2. Asymptotic form of $\mathbb{L}^\varepsilon$ acting on some kind of test functions φ^ε .
3. Singular perturbation problem: $\mathbb{L}^\varepsilon \varphi^\varepsilon = \Psi + \varepsilon \theta^\varepsilon$.

II. *Tightness:*

1. Compact containment condition

$$\lim_{M \rightarrow \infty} \sup_{0 < \varepsilon \leq \varepsilon_0} \mathbb{P} \left(\sup_{0 < t \leq T} |\xi^\varepsilon(t)| \geq M \right) = 0.$$

2. Submartingale condition. The process

$$\eta^\varepsilon(t) := \varphi(\xi^\varepsilon(t)) + C_\varphi t,$$

is a nonnegative submartingale with respect to some filtration.

In the case of semimartingales, used in proofs of Poisson approximation results, we follow the same scheme of weak convergence proof, but for their predictable characteristics presented under integral functional forms.

For Markov switching processes, step II is performed in a simpler and more adequate way by using predictable square characteristics of martingale characterization.

It is worth noticing that most of results presented in this book are new, or generalize results published previously by the authors.

The book is organized as follows.

In Chapter 1, we present shortly the basic families of stochastic processes used in this book. More specifically, we present the Markov, semi-Markov processes, and some of their subfamilies, and semimartingales.

In Chapter 2, we present the notions of switching and switched processes via additive functionals of processes with locally independent increments switched by Markov or semi-Markov processes.

In Chapter 3, we present stochastic systems in the series scheme. More specifically, we present the basic results of average and diffusion approximations; proofs are postponed until the next chapters. We start from the simplest case of integral functionals to end by the more complicated case of general additive functionals.

In Chapter 4, we present results concerning average and diffusion approximations, as in Chapter 3, but with the addition that the state space of the switching process is asymptotically merged (single and double merged). Ergodic and non-ergodic switching processes are considered.

In the next two chapters we present detailed proofs of the results stated in Chapters 3 and 4.

In Chapter 5, we present the algorithmic part of the proofs, realized by the singular perturbing reducible-invertible operator technique. This part corresponds to the finite-dimensional distribution convergence and is performed as in step I above.

In Chapter 6, we give the last step of the proofs concerning the tightness of the probability distribution of the processes. This is performed as in step II above.

In Chapter 7, we give Poisson approximation results for impulsive processes switched by Markov processes and for stochastic additive functionals switched by semi-Markov processes.

The next two chapters include applications of the theory presented in the previous chapters.

In Chapter 8, we present several applications: absorption time distributions, stationary phase merging, superposition of two independent renewal processes, semi-Markov random walks.

In Chapter 9, we present birth-and-death processes, repairable systems, and Lévy approximation of impulsive systems.

Finally, we give some problems to solve. These problems include results stated without proofs in previous chapters, additional results, and some extensions.

Three appendices are also included. The first gives some definitions about weak convergence of stochastic processes. The second gives some known basic theorems on convergence of semimartingales and composed processes needed in proofs. The third includes some additional results concerning intermediate proofs of theorems. Of course, even if these theorems are included in order that this book becomes as autonomous as possible, we encourage the readers to find the corresponding information directly to bibliography.

This book should serve as a textbook for graduate students, post-doctoral seminars or courses for applied scientists and engineers in stochastic approximation of complex systems: queuing, reliability, risk, finance,

biology systems, etc.

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Kiev
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