CHROMATIC POLYNOMIALS AND CHROMATICITY OF GRAPHS

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CHROMATIC POLYNOMIALS A N D CHROMATICITY OF GRAPHS

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Preface

For a century, one of the most famous problems in mathematics was to prove the four-colour theorem. This has spawned the development of many useful tools for solving graph colouring problems. In a paper in 1912, Birkhoff proposed a way of tackling the four-colour problem by introducing a function $P(M, \lambda)$, defined for all positive integers λ , to be the number of proper λ -colourings of a map M . It turns out that $P(M, \lambda)$ is a polynomial in λ , called the *chromatic polynomial* of M . If one could prove that $P(M, 4) > 0$ for all maps M , then this would give a positive answer to the four-colour problem. The polynomial $P(M, \lambda)$ is defined for all real and complex values of λ . It was hoped that many useful tools from algebra and analysis could be used to find or estimate the roots of the polynomial and hence lead to the resolution of the problem.

The notion of a chromatic polynomial was later generalized to that of an arbitrary graph by Whitney (1932), who established many fundamental results for it. In 1946, Birkhoff and Lewis obtained results concerning the distribution of real roots of chromatic polynomials of planar graphs and conjectured that these polynomials have no real roots greater than or equal to four. The conjecture remains open.

In 1968, Read aroused new interest in the study of chromatic polynomials with his well referenced introductory article on the subject. He asked if it is possible to find a set of necessary and sufficient algebraic conditions for a polynomial to be the chromatic polynomial of some graph. For example, it is true that the chromatic polynomial of a graph determines the numbers of vertices and edges and that its coefficients are integers which alternate in sign. Read observed that the absolute values of the coefficients appear to form a unimodal sequence.

Read asked: What is a necessary and sufficient condition for two graphs to be chromatically equivalent; that is, to have the same chromatic polynomial? In particular, Chao and Whitehead Jr. (1978) defined a graph to

be chromatically unique if no other graphs share its chromatic polynomial. They found several families of such graphs. Since then many invariants under chromatic equivalence have been found and various families of and results on such graphs have been obtained successively. The question of chromatic equivalence and uniqueness is termed the chromaticity of graphs. This remains an active area of research.

Although Birkhoff's hope of using the chromatic polynomial to prove the four-colour theorem was not borne out, it has attracted a steady stream of attention through the years, especially concerning the location of its roots. More recently, Thomassen discovered a relation between hamiltonian paths and the roots of the chromatic polynomial. There has also been an influx of new ideas from statistical mechanics due to the recent discovery of a connection to the Potts Model in Physics.

This book is divided into three main parts, after providing a chapter on the basic concepts and terminology of graphs and a list of notation that are needed and used in the book. Part one covers the first three chapters. It is devoted in greater detail than the other two to the rudiment of chromatic polynomials; their basic properties are derived, and some practical methods for computing them are given. Furthermore, we provide several ways of constructing chromatically equivalent graphs; characterize chromatically unique graphs that are disconnected and those with connectivity 1. Further results on chromatic equivalence classes of families of graphs are mentioned.

Part two, which consists of eight chapters from Chapter 4 to Chapter 11, deals specifically with the chromaticity of multi-partite graphs, subdivisions of graphs, and members of those families whose colour classes have nice structures. By expanding a chromatic polynomial of a graph in terms of falling factorials, we construct a polynomial, called the adjoint polynomial of the graph. We study several invariants of this polynomial and roots of some particular ones. It was found that this polynomial was particularly useful in determining the chromaticity of graphs whose complements are of simpler structure. We also mention some related polynomials.

The last part of the book covers the last four chapters and is concerned with the distribution of roots of the chromatic polynomials both on the real line and in the complex plane. In particular, we study those chromatic polynomials that possess only integral roots. Furthermore, we study bounds and inequalities of the chromatic polynomials of families of graphs.

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F.M. Dong, K.M. Koh and K.L. Teo
Singapore
December, 2004

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Basic Concepts in Graph Theory

1. Graphs and adjacency

A (simple) graph $G = (V(G), E(G))$ consists of a non-empty finite set $V(G)$ together with a (possibly empty) set $E(G)$ of unordered pairs of distinct elements of $V(G)$. Each element in $V(G)$ is called a *vertex* of G , and each member $\{u, v\}$ in $E(G)$, where $u, v \in V(G)$, is called an *edge* of G , which is often denoted by uv or vu . The sets $V(G)$ and $E(G)$ are called the *vertex set* and *edge set* of G respectively. The symbols $v(G)$ and $e(G)$ are used to denote the number of vertices and the number of edges in G respectively. A graph G is said to be of *order* n if $v(G) = n$ and of *size* m if $e(G) = m$. A graph of order n and size m is also called an (n, m) -graph. A graph G is said to be *trivial* if $v(G) = 1$, and non-trivial otherwise.

For instance, the graph H in Figure 1 is a $(7, 11)$ -graph with $V(H) = \{a, b, c, w, x, y, z\}$ and $E(H) = \{ac, ax, aw, bw, bx, by, bz, cy, wx, wy, xy\}$.

Figure 1

Remark By definition, any two distinct vertices of a (simple) graph are joined by at most one edge. If more than one edge is allowed to join some pairs of vertices, the resulting structure is called a *multigraph*. Unless otherwise stated, all graphs we consider in this book are simple graphs.

Let G be a graph. Two distinct vertices u and v in G are *adjacent* if uv is an edge of G . In this case, u is also called a *neighbour* of v , and vice versa. If $e = uv$ is an edge of G , we say that the edge e joins u and v , or the vertices u and v are the two *ends* of e . In this case, we also say that e is *incident* with u and v .

Let $A \subseteq V(G)$. Then A is called a *clique* in G if every two vertices in A are adjacent. On the other hand, A is called an *independent set* in G if no two vertices in A are adjacent in G . Thus for the graph H in Figure 1, the set $\{a, x, w\}$ is a clique while $\{a, b\}$ is an independent set. The *clique number* of G , denoted by $\omega(G)$, and the *independence number* of G , denoted by $\alpha(G)$, are defined as follows:

$$\omega(G) = \max\{|A| : A \text{ is a clique in } G\},$$

$$\alpha(G) = \max\{|A| : A \text{ is an independent set in } G\}.$$

Thus, in the graph H in Figure 1, we have $\omega(H) = 4$ (the largest clique being $\{b, w, x, y\}$) and $\alpha(H) = 3$ (a largest independent set being $\{c, w, z\}$).

Two edges of G are said to be *adjacent* if they have an end vertex in common. Thus, in Figure 1, ax and xy are adjacent while ac and xy are not. A set M of edges is called a *matching* in G if no two edges in M are adjacent. The matching number of G , denoted by $\alpha'(G)$, is defined as follows:

$$\alpha'(G) = \max\{|M| : M \text{ is a matching in } G\}.$$

Thus, in Figure 1, we have $\alpha'(H) = 3$ (a largest matching being $\{ac, xy, bz\}$).

2. Degrees

For $u \in V(G)$, the *neighbourhood* of u , denoted by $N_G(u)$ (or simply $N(u)$), is the set of neighbours of u ; i.e., $N(u) = \{v \in V(G) : uv \in E(G)\}$.

Let G be a multigraph and $v \in V(G)$. The *degree* of v in G , denoted by $d_G(v)$ (or simply by $d(v)$), is the number of edges of G incident with v . Thus, if G is simple, then $d(v) = |N(v)|$. A vertex v is said to be *isolated* if $d(v) = 0$; and is called an *end-vertex* if $d(v) = 1$. Assume that $V(G) = \{v_1, v_2, \dots, v_n\}$. Then the sequence $(d(v_1), d(v_2), \dots, d(v_n))$ is called a *degree sequence* of G . A simple but useful relation between the degrees of the vertices and the size of a multigraph is given below.

Hand-shaking Lemma *Let G be a multigraph of size m . Then*

$$\sum_{v \in V(G)} d(v) = 2m.$$

A vertex v in a multigraph is said to be *odd* (resp., *even*) if $d(v)$ is odd (resp., even). The following result follows readily from the above lemma.

Corollary *The number of odd vertices in any multigraph is always even.*

Given a graph G , the *maximum degree* $\Delta(G)$ and the *minimum degree* $\delta(G)$ of G are defined, respectively, as follows:

$$\begin{aligned}\Delta(G) &= \max\{d(v) : v \in V(G)\}, \\ \delta(G) &= \min\{d(v) : v \in V(G)\}.\end{aligned}$$

Thus, for the graph H in Figure 1, we have $\Delta(H) = 4$ and $\delta(H) = 1$.

A graph G is said to be *regular* if $\Delta(G) = \delta(G)$. More precisely, G is *k-regular* if $\Delta(G) = k = \delta(G)$; i.e., $d(v) = k$ for each $v \in V(G)$. A 3-regular graph is also called a *cubic* graph.

3. Isomorphic graphs and subgraphs

Two graphs G and H are said to be *isomorphic*, in notation: $G \cong H$, if there exists a bijection $\varphi : V(G) \rightarrow V(H)$ which preserves adjacency; i.e., $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$. Such a bijection φ is called an *isomorphism* of G onto H . Any isomorphism of G onto itself is called an *automorphism* of G . A graph G is said to be *vertex-transitive* (resp., *edge-transitive*) if for any two vertices u and v (resp., two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$), there exists an automorphism φ of G such that $\varphi(u) = v$ (resp., $\varphi\{u_1, v_1\} = \{u_2, v_2\}$).

Let H and G be two graphs. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; and a *proper subgraph* of G if, in addition, $H \neq G$. A subgraph H of G is said to be *spanning* if $V(H) = V(G)$. Let $A \subseteq V(G)$ and $A \neq \emptyset$. The subgraph of G *induced* by A , denoted by $G[A]$ or $[A]$, is the graph having the vertex set A and its edge set consisting of all the edges of G joining vertices in A . We call $G[A]$ the *subgraph of G induced by A* . Let $F \subseteq E(G)$ and $F \neq \emptyset$. The subgraph of G *induced* by F , denoted by $G[F]$ or $[F]$, is the graph with $E([F]) = F$ and

$$V([F]) = \{v \in V(G) : v \text{ is an end of some edge in } F\}.$$

Thus, in Figure 1, if $A = \{a, w, x, y\}$ and $F = \{aw, bw, by, bz\}$, then $[A]$ and $[F]$ are shown in Figure 2.

Figure 2

Let $A \subset V(G)$. The graph $G - A$ is the subgraph of G obtained by removing from G each vertex in A together with the edges incident with it. If $A = \{a\}$, we write $G - a$ for $G - \{a\}$. Evidently, $G - A = [V(G) \setminus A]$, which is an induced subgraph of G .

Let $F \subseteq E(G)$. The graph $G - F$ is the subgraph of G obtained by removing each edge in F from G . If $F = \{f\}$, we write $G - f$ for $G - \{f\}$. Clearly, $G - F$ is a spanning subgraph of G .

4. Connectedness

Let G be a graph. A *walk* in G is a finite sequence $v_0 v_1 v_2 \cdots v_k$ of vertices such that $v_i v_{i+1}$ is an edge for each $i = 0, 1, \dots, k-1$. The above walk is also called a $v_0 - v_k$ walk, and the *length* of the walk is k , which is the number of occurrences of its edges. Let $u, v \in V(G)$. A $u - v$ walk is said to be *closed* if $u = v$ and its length is at least two. A walk is a *trail* (resp., *path*) if no edge (resp., vertex) is repeated. Clearly, every path is a trail, but not conversely. A path is said to be *hamiltonian* (or *spanning*) if it contains all the vertices in G . A path $v_0 v_1 \cdots v_k$ is called a *chain* if $d_G(v_i) = 2$ for each $i = 1, 2, \dots, k-1$; and is a *maximal chain* if, in addition, $d_G(v_0) \geq 3$ and $d_G(v_k) \geq 3$. A *cycle* is a closed walk $v_1 v_2 \cdots v_k v_1$ in which the v_i 's are distinct. Such a cycle of order k is also called a *k-cycle*. A 3-cycle is often called a *triangle*. The *girth* of G , denoted by $g(G)$, is the order of a smallest cycle in G . A k -cycle C in G , where $k \geq 4$, is said to be *pure* if $[V(C)] \cong C$; i.e., there is no edge in G joining two non-consecutive vertices along C (such an edge, if it exists, is called a *chord* of C). Thus, in Figure 1, $g(H) = 3$, and while *axyca* is a pure cycle, the cycle *axywa* is not pure.

A graph G is *connected* if any two vertices in G are joined by a path. It is *disconnected* otherwise. A *component* of G is a connected subgraph of G which is not a proper subgraph of any connected subgraph of G . We denote by $c(G)$ the number of components of G . For any two vertices u, v

in a connected graph G , the *distance* $d_G(u, v)$ (or simply $d(u, v)$) between u and v is the length of a shortest $u - v$ path in G . The *diameter* of G , denoted by $\text{diam } G$, is defined as $\text{diam } G = \max\{d(x, y) : x, y \in V(G)\}$. Thus, in Figure 1, we have $\text{diam } H = d(z, a) = d(c, z) = 3$.

5. Special graphs

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of positive integers, and let $n \in \mathbb{N}$.

A graph is an *empty graph* if it contains no edges. The empty graph of order n is denoted by O_n . In contrast with empty graphs, a graph is said to be *complete* if any two of its vertices are adjacent. The complete graph of order n is denoted by K_n . Clearly, $e(K_n) = \binom{n}{2}$. As a graph, any n -cycle is denoted by C_n . A cycle C_n is *even* (resp., *odd*) if n is even (resp., odd). A graph is said to be *unicyclic* if it contains exactly one cycle as a subgraph.

A *forest* is a graph containing no cycles; such a graph is also said to be *acyclic*. A *tree* is a connected forest. Thus every component of a forest is a tree. It is known that for every forest F , $v(F) = e(F) + c(F)$; and in particular, for any tree T , $v(T) = e(T) + 1$. Any path of order n as a graph is denoted by P_n .

Given $q \in \mathbb{N}$, the class of q -trees is defined recursively as follows: any complete graph K_q is a q -tree, and any q -tree of order $n + 1$ is a graph obtained from a q -tree G of order n , where $n \geq q$, by adding a new vertex and joining it to each vertex of a K_q in G . A graph G is called a *chordal* graph if G contains no pure cycle (i.e., every cycle in G contains a chord). It follows that every tree is a 1-tree, and every q -tree is a chordal graph, but not conversely (see Figure 3).

Figure 3

For $t \in \mathbb{N}$ and $t \geq 2$, a t -partite graph is a graph G whose $V(G)$ can be partitioned into t non-empty subsets $V_1, V_2, \dots, V_t$ such that each edge of G joins a vertex in V_i to a vertex in V_j for some distinct i, j in $\{1, 2, \dots, t\}$.

We call $(V_1, V_2, \dots, V_t)$ a *t-partition* of G . A *complete t-partite graph* is a *t-partite graph* with a *t-partition* $(V_1, V_2, \dots, V_t)$ such that every vertex in V_i is adjacent to every vertex in V_j for all distinct i, j in $\{1, 2, \dots, t\}$. Such a complete *t-partite graph* is denoted by $K(p_1, p_2, \dots, p_t)$ if $|V_i| = p_i$ for each $i = 1, 2, \dots, t$.

A 2-partite graph is better known as a *bipartite graph*, and a 2-partition is also called a *bipartition*. Every tree is clearly bipartite and the tree $K(1, q)$ is also called a *star*. The following characterization of bipartite graphs, due to König (1916), is useful.

Theorem 1 *A graph is bipartite if and only if it contains no odd cycles.*

□

A graph is said to be *planar* if it can be drawn in the plane in such a way that no two edges intersect except at a vertex. Any such drawing is a *plane drawing* of the graph. Let G be a planar graph. Then any plane drawing of G divides the set of points of the plane into regions, called *faces*; one face is unbounded and is the *infinite face*. The beautiful Euler's formula for plane graphs G which relates the order, size and the number of faces of G is stated below.

Theorem 2 *Let G be a connected plane graph of order n and size m that has f faces. Then*

$$n - m + f = 2.$$

□

A planar graph G is said to be *outerplanar* if every vertex lies on the boundary of the same face in some plane drawing of G .

A connected multigraph G is said to be *eulerian* if there is a closed trail in G which contains each edge of G once. It is known that a connected multigraph G is eulerian if and only if every vertex of G is even.

6. New graphs formed from the given ones

Let G be a graph and $x, y \in V(G)$. We write $G \cdot xy$ to denote the graph obtained from G by contracting x and y and removing any loop and all but one of the multiple edges, if they arise. If $xy \notin E(G)$, we denote by $G + xy$ the graph obtained by adding a new edge xy to G . The *complement* of G , denoted by $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ such that two vertices are

adjacent in $\overline{G}$ if and only if they are not adjacent in G . A graph G is said to be *self-complementary* if $\overline{G} \cong G$. An edge e in G is said to be *subdivided* if it is deleted and replaced by a chain connecting its ends, the internal vertices of this chain being new. A *subdivision* of G is a graph that can be obtained from G by a finite sequence of edge subdivisions. A subdivision of G is called a *G -homeomorph*.

For $k \geq 2$, let θ_k be the multigraph with 2 vertices and k edges. Any subdivision of θ_k is called a *multi-bridge graph*. For any $a_1, a_2, \dots, a_k \in \mathbb{N}$, we denote by $\theta(a_1, a_2, \dots, a_k)$ the graph obtained by replacing the edges of θ_k with paths of lengths $a_1, a_2, \dots, a_k$ respectively. The graph $\theta(1, a_2, a_3)$ is called a θ -graph and the graph $\theta(a_1, a_2, a_3)$ is called a *generalized θ -graph*.

Let G and H be two graphs with two disjoint vertex sets. Their *disjoint union* is denoted by $G \cup H$, the disjoint union of k copies of G is denoted by kG and the disjoint union of a family of graphs $G_1, G_2, \dots, G_k$ is denoted by $\cup_{i=1}^k G_i$. The *cartesian product* of G and H , denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ such that two vertices (x, y) and (x', y') are adjacent in $G \times H$ if and only if either (i) $x = x'$ and $yy' \in E(H)$ or (ii) $y = y'$ and $xx' \in E(G)$. The cartesian product of a family of graphs $G_1, G_2, \dots, G_k$ is denoted by $\prod_{i=1}^k G_i$. The *join* of G and H , denoted by $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and

$$E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}.$$

Observe that $K(p, q) = O_p + O_q$ and, more generally,

$$K(p_1, p_2, \dots, p_t) = (\dots((O_{p_1} + O_{p_2}) + O_{p_3}) \dots) + O_{p_t}.$$

The *wheel* of order n , denoted by W_n , is defined as $W_n = C_{n-1} + K_1$. Any edge joining K_1 and a vertex in C_{n-1} is called a *spoke* of W_n . For $1 \leq s \leq n - 2$, we denote by $W(n, s)$ the graph obtained from W_n by deleting all but s consecutive spokes, as shown in Figure 4. The *fan* of order n , denoted by F_n , is defined as $F_n = P_{n-1} + K_1$.

Figure 4 $W(8, 3)$

7. Graph parameters

Let G be a connected graph of order $n \geq 1$. A proper subset S of $V(G)$ is called a *cut* of G if $G - S$ is disconnected. A vertex w is called a *cut-vertex* of G if $\{w\}$ is a cut. The *vertex-connectivity* (the prefix 'vertex' can be omitted if there is no chance of confusion) of G , denoted by $\kappa(G)$, is defined by

$$\kappa(G) = \begin{cases} n - 1, & \text{if } G \cong K_n; \\ \min\{|S| : |S| \text{ is a cut of } G\}, & \text{if } G \not\cong K_n. \end{cases}$$

We define $\kappa(G) = 0$ if G is disconnected. We say G is *k-vertex-connected* (the prefix 'vertex' can be omitted if there is no chance of confusion), where $k \in \mathbb{N}$, if $\kappa(G) \geq k$. A cut S of G is called a *clique-cut* if S is a clique in G .

Let $u, v \in V(G)$. A family $\{Q_1, Q_2, \dots, Q_k\}$ of $u - v$ paths in G is said to be *internally disjoint* if $V(Q_i) \cap V(Q_j) = \{u, v\}$ for all i, j with $1 \leq i < j \leq k$. The celebrated characterization of k -connected graphs, due to Whitney (1932) (see Bondy and Murty (1976)), is stated below.

Theorem 3 *A graph G is k -connected if and only if every two distinct vertices in G are joined by at least k internally disjoint paths.* $\square$

By definition, when $n \geq 3$, G is 2-connected if and only if G contains no cut-vertex. A *block* is a non-trivial connected graph which contains no cut-vertex. Thus G is a block if and only if $G \cong K_2$ when $n = 2$, and G is 2-connected when $n \geq 3$.

A *block* of G is a subgraph H of G such that H is itself a block, and is maximal with respect to this property. Let $b(G)$ denote the number of blocks in G . Thus, there are 9 blocks in the graph in Figure 5. An edge e in G is called a *bridge* if $G - e$ is disconnected; i.e., $\{e\}$ induces a block of G .

Figure 5

Let $k \in \mathbb{N}$. A *k-colouring* of G is a mapping

$$\varphi : V(G) \rightarrow \{1, 2, \dots, k\}$$

(not necessarily onto) such that $\varphi(u) \neq \varphi(v)$ whenever $uv \in E(G)$. A graph G is said to be k -colourable if G admits a k -colouring. The *chromatic number* of G , denoted by $\chi(G)$, is the minimum value of k such that G is k -colourable. We say G is k -chromatic if $\chi(G) = k$.

Assume that $\chi(G) = k$ with a k -colouring θ . For each $i = 1, 2, \dots, k$, let

$$V_i = \{v \in V(G) : \theta(v) = i\}.$$

We call V_i the i th *colour class* induced by θ . Clearly, each V_i is an independent set in G and $\{V_1, V_2, \dots, V_k\}$ forms a partition of $V(G)$. Thus the chromatic number of G can, equivalently, be defined as the minimum number k such that $V(G)$ is partitioned into k independent sets. The graph G is said to be *uniquely k -colourable* if any two k -colourings of G induce the same partition of $V(G)$.

The *Tuán graph* is a complete t -partite graph $K(p_1, p_2, \dots, p_t)$ such that $|p_i - p_j| \leq 1$ for all $i, j = 1, 2, \dots, t$. Such a Tuán graph of order n is denoted by $K[t; n]$. Tuán (1941) (see Bondy and Murty (1976)) proved the following result.

Theorem 4 *Let G be a graph of order n and $\omega(G) \leq k$, where $n, k \in \mathbb{N}$. Then $e(G) \leq e(K[k; n])$.* □

It follows that among the k -colourable graphs of order n , $K[k; n]$ has the maximum size.

A graph G is said to be k -critical if $\chi(G) = k$ and $\chi(H) < k$ for any proper subgraph H of G . Clearly, every k -chromatic graph contains a subgraph which is k -critical. Some useful properties of k -critical graphs are stated below.

Theorem 5 *Let G be a k -critical graph, where $k \in \mathbb{N}$. Then*

- (i) G is connected;
- (ii) $v(G) \neq k + 1$;
- (iii) $k \leq 1 + \delta(G)$;
- (iv) no cut of G is a clique;
- (v) either $G = K_k$ or $2e(G) \geq 1 + (k - 1)v(G)$.

□

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Notation

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}$$

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

$$\mathbb{R} = \text{the set of real numbers}$$

$$\mathbb{C} = \text{the set of complex numbers}$$

$$a|b : a \text{ divides } b$$

$$a \nmid b : a \text{ does not divide } b$$

$$\lfloor x \rfloor = \text{the largest integer less than or equal to } x$$

$$\lceil x \rceil = \text{the smallest integer greater than or equal to } x$$

$$a \equiv b \pmod{k} : a \text{ is congruent to } b \text{ modulo } k; \text{ i.e., } k|(a - b)$$

$$|S| = \text{the number of elements in the finite set } S$$

$$s(r, n) = (\text{signed}) \text{ Stirling number of the first kind}$$

$$S(r, n) = \text{Stirling number of the second kind}$$

$$\binom{n}{r} = \text{the number of } r\text{-element subsets of an } n\text{-element set} = \frac{n!}{r!(n-r)!}$$

$$(\lambda)_n = \lambda(\lambda - 1) \cdots (\lambda - n + 1)$$

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

$$B \setminus A = \{x \in B : x \notin A\}$$

In what follows, G and H are graphs.

$$V(G) : \text{the vertex set of } G$$

$$E(G) : \text{the edge set of } G$$

$$v(G) : \text{the number of vertices in } G \text{ or the order of } G$$

$$e(G) : \text{the number of edges in } G \text{ or the size of } G$$

$$N(u) = N_G(u) : \text{the set of vertices } v \text{ such that } uv \in E(G)$$

$$\overline{G} : \text{the complement of } G$$

- $[A]$: the subgraph of G induced by A , where $A \subseteq V(G)$
 $[F]$: the subgraph of G induced by F , where $F \subseteq E(G)$
 $G - v$: the subgraph of G obtained by removing v and all edges incident with v from G , where $v \in V(G)$
 $G - e$: the subgraph of G obtained by removing e from G , where $e \in E(G)$
 $G - F$: the subgraph of G obtained by removing all edges in F from G , where $F \subseteq E(G)$
 $G - A$: the subgraph of G obtained by removing each vertex in A together with the edges incident with vertices in A from G , where $A \subseteq V(G)$
 $G \cdot xy$: the graph obtained from G by contracting x and y and removing any loop and all but one of the multiple edges, if they arise, where $x, y \in V(G)$
 $G + xy$: the graph obtained by adding a new edge xy to G , where $x, y \in V(G)$ and $xy \notin E(G)$
 $\mathcal{G}[G_1 \cup_r G_2]$: the family of all K_r -gluings of G_1 and G_2
 $d(v) = d_G(v)$: the degree of v in G , where $v \in V(G)$
 $d(u, v) = d_G(u, v)$: the distance between u and v in G , where $u, v \in V(G)$
 $\text{diam } G = \max\{d(x, y) : x, y \in V(G)\}$
 $c(G)$: the number of components in G
 $b(G)$: the number of blocks in G
 $g(G)$: the girth of G
 $r(G) = v(G) - c(G)$, the rank of G
 $r_c(G) = e(G) - v(G) + c(G)$, the cyclomatic number (or nullity) of G
 $t(G)$: the number of triangles in G
 $\delta(G)$: the minimum degree of G
 $\Delta(G)$: the maximum degree of G
 $\alpha(G)$: the independence number of G
 $\alpha'(G)$: the matching number of G
 $\alpha(G, k)$: the number of partitions of $V(G)$ into k non-empty independent sets
 $\kappa(G)$: the connectivity of G
 $\omega(G)$: the clique number of G

$\chi(G)$: the chromatic number of G

$G + H$: the join of G and H

$G \cup H$: the disjoint union of G and H

kG : the disjoint union of k copies of G

$G \times H$: the cartesian product of G and H

$P(G, \lambda)$: the chromatic polynomial of G

$G \sim H$: G and H are χ -equivalent; i.e., $P(G, \lambda) = P(H, \lambda)$

$s_r(G) = P(G, r)/r!$

$n_G(H)$: the number of subgraphs in G which are isomorphic to H

$i_G(H)$: the number of induced subgraphs in G which are isomorphic to H

C_n : the cycle of order n

F_n : the fan of order n

K_n : the complete graph of order n

O_n : the empty graph of order n

P_n : the path of order n

W_n : the wheel of order n

$K(p, q)$: the complete bipartite graph with a bipartition (V_1, V_2) such that $|V_1| = p$ and $|V_2| = q$

$K(p_1, p_2, \dots, p_t)$: the complete t -partite graph with a t -partition $(V_1, V_2, \dots, V_t)$ such that $|V_i| = p_i$ for $i = 1, 2, \dots, t$

$[G] = \{H : H \sim G\}$

$K[t; n]$: the Turán graph of order n and clique number t

$W(n, k)$: the subgraph of W_n obtained by deleting all but k consecutive spokes

$\theta(a_1, a_2, \dots, a_k)$: the multi-bridge graph obtained by joining two distinct vertices by k internally-disjoint paths of lengths $a_1, a_2, \dots, a_k$ respectively

$\theta_k(f)$: a special multi-bridge graph $\theta(a_1, a_2, \dots, a_k)$, where $a_1 = a_2 = \dots = a_k = f$