AN INNOVATION Application of **APPROACH TO** White Noise Theory **RANDOM FIELDS**

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Takeyuki Hida

Meijo University, Japan

Si Si

Aichi Prefectural University, Japan

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— Application of White Noise Theory

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To the memory of Fumi and Su Su

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Preface

The purpose of this book is two-fold: First, we analyze random fields in line with *innovation theory*. The analysis will be carried out as an application of white noise analysis which has developed extensively in recent years. Most of the random fields to be discussed are complex systems that are parameterized by a space-time parameter t or by a manifold C like a curve and surface and are evolutionary with respect to those parameters. In what we discuss here, one can see, among the highlights of this volume, the effectiveness of the variational calculus applied to the random fields.

Our second aim is to show interesting *applications* in many areas, such as quantum dynamics, quantum field theory, statistical mechanics and molecular biology, as well as many other fields of science. There one can see significance of the innovation approach to random complex systems that are developing as the parameter varies.

The random fields that we discuss in this volume are assumed to be functionals of white noise of either Gaussian or Poisson type, parameterized by a multi-dimensional time (vector) t or by a certain manifold C that is running through a Euclidean space. Let a field be denoted by $X(t) = X(t, x)$ or by $X(C) = X(C, x)$, where x is a sample function of a white noise and t is a vector in a certain topological vector space, and where C changes within a certain family $\mathbf{C}$ of smooth, closed and convex manifolds in R^d , $d \geq 1$. We focus our attention mainly on $X(C)$, where, in particular, C is chosen so as to run through a space-time domain. Thus, $X(C)$ can be thought of as a realization of random complex phenomena varying as C deforms and being interfered with by a fluctuation, a mathematical expression of which is a white noise x .

Our main interest is the determination of the probabilistic structure of $X(C)$. There are several mathematical tools for this aim; among others, we first provide its stochastic variation, denoted by $X(C)$, obtained by an

infinitesimal deformation of the C . The variation therefore expresses the infinitesimal change of the random phenomenon in question. Further if we can introduce an equation for stochastic variation, then it (or its solution) characterizes the structure of $X(C)$. The study of such variational calculus is to be a sort of stochastic calculus. It might be considered an analogue of the classical theory of calculus of variation. In fact, it is partially true, but not quite. In addition to the classical results, the stochastic part of the variational calculus plays a more important role, and its study requires some advanced theory of probability.

We shall also discuss random functions depending on a function f . There are many good examples in these cases. It is noted that some $X(C)$ discussed above can often be reduced to this case, since C may be represented by a vector-valued function.

With these assumptions, we are now ready to provide some background including the following topics which are known concepts in white noise theory:

- (1) White noise with multi-dimensional parameters and with its restriction to a lower dimensional manifold. Similar facts for Poisson noise.
- (2) Generalized functionals of white noise, and the S -transform that carries those functionals to non-random functionals of ordinary real valued functions.
- (3) Theory of calculus of variations that provides a powerful tool in the analysis of random fields with the help of the S -transform.
- (4) Infinite dimensional rotation group that describes the invariance of the white noise measure.

With this background we shall discuss **stochastic variational calculus** and its applications. We can see various applications that suggest to us new directions of our white noise analysis, in particular directions to quantum dynamics and quantum computation.

Before we come to the actual steps of the main subjects, we first need to determine the topology to be introduced in the class of random functions and stochastic processes, so that the variation is rigorously defined. The possible topologies are as follows:

- (i) The mean square topology. In this case we start with a Hilbert space involving square integrable functionals with respect to the white noise measure. The norm of the space is given by the positive square root of the quadratic mean. The Gaussian fields have been extensively investigated within this framework. Given either Gaussian noise or other

noise, the so called S -transform, mentioned in (2) above of random functional of noise is applied, so that we can use a modern version of the theory of functional analysis.

- (ii) The convergence in probability. Since there are cases where the existence of moments of the random variables in question is not required, we want to discuss a sufficiently wide class of random functions, in particular random fields. Path-wise nonlinear operations are naturally introduced and discussed there. The use of characteristic functional is efficient in this case.

We are now in a position to speak of the most *significant characteristics* as well as the central *advantages* of our approach, although space is too limited to describe the details. They are listed below.

- (1) If one is allowed to use intuitive notations of white noise and Poisson noise like $\dot{B}(t)$ and $\dot{P}(t)$, it is claimed that they are taken to be *variables* of random functions that describe the given random complex systems. As a result, we are led to introduce generalized functionals of noise. This allows us to discuss stochastic analysis for a sufficiently broad applicable area. Further, it is natural to introduce differential and integral operators in $\dot{B}(t)$ and $\dot{P}(t)$ and to establish an analysis of new kind.
- (2) White noise measure enjoys rich invariance, so that the infinite dimensional rotation group can be introduced (see (4) above). This enables us to discuss a *Harmonic Infinite Dimensional Analysis*. Note that the analysis is essentially infinitely dimensional, the exact meaning of which will be illustrated in this book.
- (3) It is emphasized that both $\dot{B}(t)$ and $\dot{P}(t)$ are elemental, being members of idealized innovation. With the factor dt they can serve as random measures. On the other hand, their sample functions (generalized functions) as well as functionals of them can be dealt with within our framework. Sample function-wise stochastic properties can be investigated by this idea. For detailed calculus, we introduce a wider class of those functionals, denoted by $(\mathbf{P})$, without assuming the existence of variances. Precise characters of *space-time dependency* and *causality* of evolutionary phenomena can now be discussed in a natural manner.
- (4) *Qualitative* probabilistic properties, like stochastic optimality and reversibility, can smoothly be discussed within our framework. In particular, one can see more clearly in the case of random fields. Further,

our setup enables us to see good connections to *quantum probability* and to *quantum dynamics*.

- (5) Of course there are many applications where a person can receive a new insight and can see beautiful interplay between mathematics and other fields of science. The reader might think that this book aims mainly at some applications of mathematics, but that is not quite true. Application of mathematics only contributes little to mathematics. On the other hand, we know that applications of physics or of others to mathematics have often made vital contributions to mathematics. Part of this idea can be seen in this volume.

A final note is that most of the parts of this book are easily understood, and we may say that the book is elemental, but not elementary. However, occasionally some more difficult ideas ground this work, as the readers might guess. Thus, this book can be read on different levels, depending on the reader.

It is our regret that we did not make good connections with non-commutative geometry or some thoughts on topics related to path integrals, which can be discussed from our viewpoint. Readers will find a short note on this topic in the Epilogue at the end of this volume. Some supplementary notes related to this topic are also in the Epilogue.

T. Hida and Si Si
July 2003, Nagoya, Japan

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