The Stress–Strength Model and its Generalizations

Theory and Applications

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The diagram illustrates the Stress–Strength Model. It features a large right-pointing arrow on the left, representing stress, and four concentric, slightly curved vertical lines on the right, representing strength. Below these symbols is the mathematical expression $P(X < Y)$, which represents the probability that the stress X is less than the strength Y .

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**THE STRESS-STRENGTH MODEL AND ITS GENERALIZATIONS
THEORY AND APPLICATIONS**

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To our families

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Preface

The term “stress” has acquired in the second half of the 20-th century a special meaning to a modern person. We all are continuously under stress, and, alas, not always have the “strength” to overcome it. The stress-strength relationship is nowadays studied in many branches of science such as psychology, medicine, pedagogy, etc. and the pharmaceutical industry accumulates billion-dollar profits assisting us to overcome or at least alleviate psychological stresses.

Broadly speaking, the term stress is used nowadays in two different meanings: 1) structural, mechanical (or engineering) stress studied in the engineering discipline called “strength of materials”, and more a more recent concept of 2) psychological stress which is usually defined as any “external stimulus – from threatening words to the sound of gunshot – which the brain interprets as dangerous”. Another way of describing this concept is “a demand, threat or other event that requires an individual to cope with a charged situation.”

In engineering, stress in a solid body (liquids do not admit engineering stress) arises due to applied loads and is defined as “the force per unit of area (pounds per square inch, say) that one part of the body exerts on adjacent parts”. Some knowledge of strength of materials existed already when ancient Greeks and Romans built their large structures, but the modern science began at the time of Galileo Galilei (1564–1642) who was one of the first of a long line of mathematicians, physicists and engineers to constitute and develop this field. The subject is being now researched in engineering schools, commercial, industrial and government supported laboratories and various societies such as the earliest Society for Experimental Stress Analy-

sis organized in 1943 which together with other institutions publicizes and disseminates the methodology and applications.

Psychological stress is a concept of a more recent origin stimulated by the work of Sigmund Freud (1856–1939) – the father of psychoanalysis. More specifically, Canadian scientist Hans Selye pioneered stress research in 1926 by investigating the effects of stress on the body. Somewhat later on, two American psychiatrists Thomas H. Holmes and Richard Bake have devised the stress measuring methodology ranking 43 critical stresses according to severity of their impact on an individual life. Since then the investigations of various aspects of psychological stress is one the most vigorous and lucrative fields of modern psychology.

The book that you are holding now in your hands has however little to do with these activities. It is devoted to a seemingly very simple topic – estimation of the probabilities of the type $P(X < Y)$, $P(X < Y < Z)$, etc., and their extensions. However, in spite of their apparent simplicity, this set of probabilistic models, usually referred as the “stress-strength” models, are of substantial interest and usefulness in various subareas of engineering (most prominently, in reliability theory), psychology, genetics, clinical trials, not to mention purely statistical problems connected with the well-known Mann-Whitney tests. The theory is often far from elementary, and applications may be often subtle and non-trivial.

In the course of working on the project we have discovered that the book about various aspects of stress-strength problem in the above described sense is long overdue. The list of references includes almost three hundred books and papers more than two thirds of which are intrinsically related to the topic. They have been produced by authors from various parts of the world who are conducting their research often oblivious of the results obtained by their fellow-scientists on the other parts of the globe. Moreover, papers devoted to this subject matter are spread across diverse disciplines (statistics, engineering, reliability, quality control, medicine, etc.), many of them being published in specialized journals. Hence, there is no database which will supply an interested individual with even a minor part of the references presented in this book. It is quite surprising and even disappointing that NO book summarizing the results of investigations on the stress-strength models has been published until now. To the best of our knowledge, the only review publications are the monograph *Statistical Estimation Problems in “Stress-Strength” Models* by V. Ivshin and Y. Lumelskii (1995) and the review article by R.A.Johnson “Stress-strength

Models for Reliability" (1988) in *Handbook of Statistics*, Vol. 7. The book by Ivshin and Lumelskii (1995) (which serves as a blueprint and impetus to our volume) was written in Russian and published by the Perm University Press being unavailable and thus unknown to the Western readers. It is devoted mainly to the point estimation of $P(X < Y)$ focusing on the best unbiased and the maximum likelihood estimation techniques. Interval estimation of $P(X < Y)$ is studied only in the case of the normal distribution, nonparametric methods of estimation are confined to just five pages and Bayes techniques are completely left out of the book. The Johnson's article, although more balanced, is only twenty eight pages long which makes it impossible to cover the diverse approaches and treatments of the stress-strength models that has been accumulated in the last five decades since their inception in the middle of the 20-th century.

The objective of our book is to attempt to fill in this void and provide a relatively comprehensive treatment of the stress-strength models. Nevertheless, we have tried to keep the level of this book as elementary as possible. Basic knowledge of undergraduate-level mathematics and statistics is a prerequisite, and many basic concepts are explained in the book with the aim to keep expositions as self-contained as possible. In Chapter 1 we acquaint the reader with the various aspects of the stress-strength models, its history, applications and extensions. Chapter 2 presents the necessary background knowledge describing statistical techniques associated with estimation of $R = P(X < Y)$. Chapter 3 is devoted to various parametric methods of point estimation of R and some of its multivariate extensions. In this chapter we derive analytical expressions for estimators when X and Y belong to "standard" parametric families. Chapter 4 deals with various parametric techniques of interval estimation of $R = P(X < Y)$. Chapter 5 is reserved for nonparametric methods of the point and interval estimation of R . Chapter 6 concentrates on major extensions and some special cases of the stress-strength models. Finally, Chapter 7 surveys applications which we were able to acquire from scanning various diverse sources.

The book thus can be viewed as the initial exhaustive review monograph on a subject of interest and importance for both theoretically-oriented scientists and practitioners. It can be used as a textbook for a low graduate/upper undergraduate level special topics courses in reliability theory or statistical methodology in general. Each chapter is supplemented by a number of problems. The book as a whole can serve as a rich and com-

prehensive source for research projects at the undergraduate and graduate levels for students in statistics and engineering: to locate “blank spots” in the research conducted so far all is needed is to casually leaf through the chapters. At last, this book is consolidation of research of scientists all over the world, from the USA, Canada and Russia, India and Korea, Japan, Eastern Europe and Israel who have worked on the problems related to the stress-strength models. We are indebted to many authors (too numerous to be listed here individually) who supplied us with copies of the valuable advise of their contributions and to Professor Norman L. Johnson with whom we exchanged ideas on this topics especially in the early stages of the project.

We hope that you will enjoy and profit from this outcome of our efforts.
S.K. Washington, D.C., USA
Y.L. Haifa, Israel
M.P. Orlando, Florida, USA
September 2002

Some Notations and Abbreviations

pdf	probability density function
cdf	cumulative distribution function
edf	empirical distribution function
MLE	maximum likelihood estimation (estimator)
LRT	likelihood ratio test
MSE	mean squared error
EB	empirical Bayes
UMVUE	uniformly minimum variance unbiased estimator
HPD region	highest probability density (region)
BVED	bivariate exponential distribution
WMW statistic	Wilcoxon-Mann-Whitney statistic
ROC curve	receiver operating characteristic curve
OD graph	ordinal dominance graph
$\hat{R}$	the MLE of R
$\hat{R}$	the UMVUE of R
$\check{R}$	the Bayes estimator of R
$\underline{X}$	$(X_1, \dots, X_{n_1})$
$\underline{Y}$	$(Y_1, \dots, Y_{n_2})$
$\sim$	distributed as
$\dot{\sim}$	approximately distributed as
$\Phi(\cdot)$	the cdf of the standard normal distribution
$\Gamma(\cdot)$	the Gamma function
$B(\cdot, \cdot)$	the Beta function

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