TOPICS IN THE THEORY OF **SCHRÖDINGER OPERATORS**

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editors

Huzihiro Araki

Kyoto University, Japan

Hiroshi Ezawa

Gakushuin University, Japan

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PREFACE

The aim of this book is to give reviews accessible to graduate students and non-expert researchers on some of the recent topics in the theory of Schrödinger operators.

The article of Yajima is a survey on the theory of the Schrödinger equation when the potential depends on the time periodically, on which Yajima is a leading expert. This article is recommended not only for young students who want to enter into this field but also to researchers who want to know recent results on this subject.

The article of Martinez, Nakamura and Sordani presents a good and practical introduction to the so-called FBI transformation (also called coherent state expansion in physics), explaining its application to the semi-classical limit of the S -matrix for Schrödinger operators. This article also serves as an introduction to mathematical analysis based on pseudo-differential (or micro-local) calculus, which is a recent trend.

The article of Isozaki is an overview of inverse spectral and scattering problems. It deals with inverse boundary value problems in 1-dimension, isospectral problems from differential geometry, and multi-dimensional inverse scattering problems for Schrödinger operators. In particular it contains a detailed exposition of Faddeev's theory. It is a readable survey from a broad point of view with a rich reference to relevant literature and is recommended as a good introduction to this subject.

The article of Hiroshima treats the Pauli-Fierz model where a non-relativistic particle interacts with the radiation fields, the latter being described on a Fock space. In particular, the ground state is studied in detail, including its existence proof and information about its charge density. The mathematical methods used in the analysis are explained in detail starting from the beginning, a special feature being the method of functional integral.

A short introduction on elementary aspects of the theory of Schrödinger

operators and relevant terminologies are given with a list of textbook references.

The four longer articles are based on lectures given at the Summer School “Mathematical Physics 2000 — Mathematical aspects of quantum mechanics”. It is one in the series of annual summer schools organized by the editors of this book since 1987 and held at Gakushuin University, for the purpose of providing easily accessible lectures on various topics on Mathematical Physics for graduate students as well as for researchers (non-experts in the field chosen for that year) wishing to widen their scopes. The particular summer school in 2000 was organized in memory of late Professor Tosio Kato, one of the founders of the subject.

In conclusion, we express our gratitude to all four authors for their excellent lecture notes.

August, 2003

Huzihiro Araki
(Professor Emeritus of Kyoto University)
Hiroshi Ezawa
(Professor Emeritus of Gakushuin University)

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