A Guide for Students and Researchers

Statistical Mechanics made Simple

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Statistical Mechanics made Simple

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Contents

Preface ix

Introduction: Theories of Thermodynamics, Kinetic Theory and
Statistical Mechanics xiii

**Chapter 1 Elementary Concepts in Statistics
and Probability 1**

1.1. The Binomial Distribution 1

1.2. Length of a Winning Streak 3

1.3. Brownian Motion and the Random Walk 4

1.4. Poisson versus Normal (Gaussian) Distributions 5

1.5. Central Limit Theorem (CLT) 9

1.6. Multinomial Distributions, Statistical Thermodynamics 12

1.7. The Barometer Equation 14

1.8. Other Distributions 14

Chapter 2 The Ising Model and the Lattice Gas 17

2.1. Some Background and Motivation 17

2.2. First-Principles Statistical Theory of Paramagnetism 18

2.3. More on Entropy and Energy 21

2.4. Some Other Relevant Thermodynamic Functions 21

2.5. Mean-Field Theory, Stable and Metastable Solutions 23

2.6. The Lattice Gas 26

2.7. The Nearest-Neighbor Chain: Thermodynamics in 1D 26

2.8. The Disordered Ising Chain 28

2.9. Other Magnetic Systems in One Dimension 28

Chapter 3	Elements of Thermodynamics	31
3.1.	The Scope of Thermodynamics	31
3.2.	Equations of State and Some Definitions	32
3.3.	Maxwell Relations	35
3.4.	Three Important Laws of Thermodynamics	35
3.5.	The Second Derivatives of the Free Energy	38
3.6.	Phase Diagrams for the van der Waals Gas	39
3.7.	Clausius–Clapeyron Equation	43
3.8.	Phase Transitions	46
3.9.	The Carnot Cycle	49
3.10.	Superconductivity	51
Chapter 4	Statistical Mechanics	55
4.1.	The Formalism — and a False Start	55
4.2.	Gibbs’ Paradox and Its Remedy	58
4.3.	The Gibbs Factor	59
4.4.	The Grand Ensemble	60
4.5.	Non-Ideal Gas and the 2-Body Correlation Function	62
4.6.	The Virial Equation of State	64
4.7.	Weakly Non-Ideal Gas	65
4.8.	Two-body Correlations	68
4.9.	Configurational Partition Function in 1D	73
4.10.	One Dimension versus Two	75
4.11.	Two Dimensions versus Three: The Debye–Waller Factors	77
Chapter 5	The World of Bosons	83
5.1.	Two Types of Bosons and Their Operators	83
5.2.	Number Representation and the Many-Body Problem	86
5.3.	The Adiabatic Process and Conservation of Entropy	88
5.4.	Many-Body Perturbations	89
5.5.	Photons	90
5.6.	Phonons	93
5.7.	Ferromagnons	96
5.8.	Conserved Bosons and the Ideal Bose Gas	99
5.9.	Nature of “Ideal” Bose–Einstein Condensation	101
5.10.	Ideal Bose-Einstein Condensation in Low Dimensions	104
5.11.	Consequences of a Hard Core Repulsion in 1D	106
5.12.	Bosons in 3D Subject to Weak Two-Body Forces	109
5.13.	Superfluid Helium (He II)	114

Chapter 6 All About Fermions: Theories of Metals, Superconductors, Semiconductors 119

6.1.	Fermi–Dirac Particles	119
6.2.	Slater Determinant: The Ground State	120
6.3.	Ideal Fermi–Dirac Gas	121
6.4.	Ideal Fermi–Dirac Gas with Spin	123
6.5.	Fermi Integrals	124
6.6.	Thermodynamic Functions of an Ideal Metal	125
6.7.	Quasiparticles and Elementary Excitations	128
6.8.	Semiconductor Physics: Electrons and Holes	130
6.9.	n -Type Semiconductor Physics: The Statistics	131
6.10.	Correlations and the Coulomb Repulsion	132
6.11.	Miscellaneous Properties of Semiconductors	135
6.12.	Aspects of Superconductivity: Cooper Pairs	137
6.13.	Aspects of BCS Theory	140
6.14.	Contemporary Developments in Superconductivity	146

Chapter 7 Kinetic Theory 149

7.1.	Scope of This Chapter	149
7.2.	Quasi-Equilibrium Flows and the Second Law	150
7.3.	The Collision Integral	151
7.4.	Approach to Equilibrium of a “Classical” Non-Ideal Gas	154
7.5.	A New Look at “Quantum Statistics”	156
7.6.	Master Equation: Application to Radioactive Decay	158
7.7.	Boltzmann Equation	160
7.8.	Electrical Currents in a Low-Density Electron Gas	162
7.9.	Diffusion and the Einstein Relation	165
7.10.	Electrical Conductivity of Metals	165
7.11.	Exactly Solved “Backscattering” Model	166
7.12.	Electron-Phonon Scattering	169
7.13.	Approximating the Boltzmann Equation	170
7.14.	Crossed Electric and Magnetic Fields	171
7.15.	Propagation of Sound Waves in Fluids	173
7.16.	The Calculations and Their Result	178

Chapter 8 The Transfer Matrix 183

8.1.	The Transfer Matrix and the Thermal Zipper	183
8.2.	Opening and Closing a “Zipper Ladder” or Polymer	186
8.3.	The Full Zipper ($N > 2$)	190

8.4.	The Transfer Matrix and Gaussian Potentials	191
8.5.	Transfer Matrix in the Ising Model	192
8.6.	The Ising Ladder or Polymer	195
8.7.	Ising Model on the Isotropic Square Lattice (2D)	196
8.8.	The Phase Transition	201
8.9.	A Question of Long-Range Order	202
8.10.	Ising Model in 2D and 3D	204
8.11.	Antiferromagnetism and Frustration	206
8.12.	Maximal Frustration	208
8.13.	Separable Model Spin-Glass without Frustration	210
8.14.	Critical Phenomena and Critical Exponents	212
8.15.	Potts Models	215

Chapter 9	Some Uses of Quantum Field Theory in Statistical Physics	219
9.1.	Outline of the Chapter	219
9.2.	Diffusion on a Lattice: Standard Formulation	219
9.3.	Diffusion as Expressed in QFT	222
9.4.	Diffusion <i>plus</i> One-Body Recombination Processes	225
9.5.	Diffusion and Two-Body Recombination Processes	226
9.6.	Questions Concerning Long-Range Order	228
9.7.	Mermin–Wagner Theorem	230
9.8.	Proof of Bogolubov Inequality	233
9.9.	Correlation Functions and the Free Energy	234
9.10.	Introduction to Thermodynamic Green’s Functions	237
	Index	245

Preface

I dedicate this book to those generations of students who suffered through endless revisions of my class notes in statistical mechanics and, through their class participation, homework and projects, helped shape the material.

My own undergraduate experience in thermodynamics and statistical mechanics, a half-century ago at MIT, consisted of a single semester of Sears' *Thermodynamics* (skillfully taught by the man himself.) But it was a subject that seemed as distant from "real" physics as did poetry or French literature. Graduate study at the University of Illinois in Urbana-Champaign was not that different, except that the course in statistical mechanics was taught by the brilliant lecturer Francis Low the year before he departed for... MIT. I asked my classmate J.R. Schrieffer, who presciently had enrolled in that class, whether I should chance it later with a different instructor. He said not to bother — that he could explain all I needed to know about this topic over lunch.

On a paper napkin, Bob wrote " $e^{-\beta H}$ ". "That's it in a nutshell!" "Surely you must be kidding, Mr Schrieffer," I replied (or words to that effect.) "How could you get the Fermi-Dirac distribution out of THAT? "Easy as pie," was the reply^a... and I was hooked.

I never did take the course, but in those long gone days it was still possible to earn a Ph.D. without much of a formal education. Schrieffer, of course, with John Bardeen and Leon Cooper, went on to solve the statistical mechanics of superconductors and thereby earn the Nobel prize.

The standard book on statistical physics in the late 1950's was by T. L. Hill. It was recondite but formal and dry. In speaking of a different text that was feebly attempting the same topic, a wit quipped that "it was not worth

^aSee Chapter 6.

a bean of Hill's." Today there are dozens of texts on the subject. Why add one more?

In the early 1960's, while researching the theory of magnetism, I determined to understand the two-dimensional Ising model that had been so brilliantly resolved by Lars Onsager, to the total and utter incomprehension of just about everyone else. Ultimately, with the help of Elliot Lieb and Ted Schultz (then my colleagues at IBM's research laboratory,) I managed to do so and we published a reasonably intelligible explanation in *Reviews of Modern Physics*. This longish work — parts of which appear in Chapter 8 — received an honorable mention almost 20 years later, in the 1982 Nobel lecture by Kenneth G. Wilson, who wrote:

"In the summer of 1966 I spent a long time at Aspen. While there I carried out a promise I had made to myself while a graduate student, namely [to work] through Onsager's solution of the two-dimensional Ising model. I read it in translation, studying the field-theoretic form given in Lieb, Mattis and Schultz [s paper.] When I entered graduate school I had carried out the instructions given to me by my father and had knocked on both Murray Gell-Mann's and [Richard] Feynman's doors and asked them what they were currently doing. Murray wrote down the partition function of the three-dimensional Ising model and said it would be nice if I could solve it.... Feynman's answer was "nothing." Later, Jon Mathews explained some of Feynman's tricks for reproducing the solution for the two-dimensional Ising model. I didn't follow what Jon was saying, but that was when I made my promise.... As I worked through the paper of Mattis, Lieb and Schultz I realized there should be applications of my renormalization group ideas to critical phenomena..."^b

Recently, G. Emch has reminded me that at the very moment Wilson was studying our version of the two-dimensional Ising model I was attending a large IUPAP meeting in Copenhagen on the foundations and applications of statistical mechanics. My talk had been advertised as, "The exact solution of the Ising model in three dimensions" and, needless to say, it was well attended. I did preface it by admitting there was no exact solution but that — had the airplane taking me to Denmark crashed — the title alone would have earned me a legacy worthy of Fermat. Although it was anticlimactic, the actual talk^c demonstrated that in 5 spatial dimensions or higher, mean-field theory prevails.

^bFrom *Nobel Lectures in Physics (1981–1990)*, published by World Scientific.

^cIt appeared in the Proceedings with a more modest title befitting a respectable albeit approximate analysis.

In the present book I have set down numerous other topics and techniques, much received wisdom and a few original ideas to add to the “hill of beans.” Whether old or new, all of it *can* be turned to advantage. My greatest satisfaction will be that you read it here first.

D.C.M.
Salt Lake City, 2003

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Introduction: Theories of Thermodynamics, Kinetic Theory and Statistical Mechanics

Despite the lack of a reliable atomic theory of matter, the science of Thermodynamics flourished in the 19th Century. Among the famous thinkers it attracted, one notes William Thomson (Lord Kelvin) after whom the temperature scale is named, and James Clerk Maxwell. The latter's many contributions include the "distribution function" and some very useful differential "relations" among thermodynamic quantities (as distinguished from his even more famous "equations" in electrodynamics). The Maxwell relations set the stage for our present view of thermodynamics as a science based on function theory while grounded in experimental observations.

The kinetic theory of gases came to be the next conceptual step. Among pioneers in this discipline one counts several unrecognized geniuses, such as J. J. Waterston who — thanks to Lord Rayleigh — received posthumous honors from the very same Royal Society that had steadfastly refused to publish his works during his lifetime. Ludwig Boltzmann committed suicide on September 5, 1906, depressed — it is said — by the utter rejection of his atomistic theory by such colleagues as Mach and Ostwald. Paul Ehrenfest, another great innovator, died by his own hand in 1933. Among 20th Century scientists in this field, a sizable number have met equally untimely ends. So "*now*", (here we quote from a well-known and popular text^a) "*it is our turn to study statistical mechanics*".

The postulational science of Statistical Mechanics — originally introduced to justify and extend the conclusions of thermodynamics but nowadays extensively studied and used on its own merits — is entirely a product of the 20th Century. Its founding fathers include Albert Einstein (who, among his many other contributions, made sense out of Planck's Law) and J. W. Gibbs, whose formulations of phase space and entropy basically

^aD. H. Goodstein, *States of Matter*, Dover, New York, 1985.

anticipated quantum mechanics. Many of the pioneers of quantum theory also contributed to statistical mechanics. We recognize this implicitly whenever we specify particles that satisfy “Fermi–Dirac” or “Bose–Einstein” statistics, or when we solve the “Bloch equation” for the density matrix, or when evaluating a partition function using a “Feynman path integral”.

In its most simplistic reduction, *thermodynamics* is the study of mathematical identities involving partial derivatives of well defined functions. These relate various macroscopic properties of matter: pressure, temperature, density, magnetization, etc., to one another. Phase transitions mark the discontinuities of one or more of these functions and serve to separate distinct regions (e.g. vapor from solid) in the variables’ phase space. *Kinetic theory* seeks to integrate the equations of motion of a many-body system starting from random initial conditions, thereby to construct the system’s thermodynamic properties. Finally, *statistical mechanics* provides an axiomatic foundation for the preceding while allowing a wide choice of convenient calculational schemes.

There is no net flow of matter nor of charged particles in thermodynamic equilibrium. Away from equilibrium but in or near *steady state*, the *Boltzmann equation* (and its quantum generalizations by Kubo and others) seeks to combine kinetic theory with statistical mechanics. This becomes necessary in order to explain and predict transport phenomena in a non-ideal medium, or to understand the evolution to equilibrium when starting from some arbitrary initial conditions. It is one of the topics covered in the present text.

Any meaningful approach revolves about taking N , the number of distinct particles under consideration, to the limit $N \rightarrow \infty$. This is not such a dim idea in light of the fact that Avogadro’s number, $N_A = 6.022045 \times 10^{23}$ per mole.^b

Taking advantage of the simplifications brought about by the law of large numbers and of some 18th Century mathematics one derives the underpinnings for a science of statistical mechanics and, ultimately, finds a theoretical justification for some of the dogmas of thermodynamics. In the 9 chapters to follow we see that a number of approximate relations at small values of N become exact in the “*thermodynamic limit*” (as the procedure of taking the

^bA *mole* is the amount of a substance that contains as many elementary entities as there are carbon atoms in 12 gm of Carbon 12. E.g.: 1 mole of electrons (e^-) consists of N_A particles of total mass 5.4860×10^{-4} gm and total charge -96.49×10^3 coulombs.

limit $N \rightarrow \infty$ is now known in all branches of physics, including many-body physics and quantum field theory).

Additionally we shall study the *fluctuations* $O(\sqrt{N})$ in macroscopic $O(N)$ *extensive* quantities, for “one person’s noise is another person’s signal”. Even when fluctuations are small, what matters most is their relation to other thermodynamic functions. For example, the “noise” in the internal energy, $\langle E^2 \rangle - \langle E \rangle^2$, is related to the same system’s heat capacity dE/dT . Additional examples come under the rubric of the “fluctuation-dissipation” theorem.

With Bose–Einstein condensation, “high”-temperature superconductivity, “nanophysics”, “quantum dots”, and “colossal” magnetoresistance being the order of the day, there is no lack of contemporary applications for the methods of statistical physics. However, first things first. We start the exposition by laying out and motivating the fundamentals and methodologies that have “worked” in such classic systems as magnetism and the non-ideal gas. Once mastered, these reductions should allow one to pose more contemporary questions. With the aid of newest techniques — some of which are borrowed from quantum theory — one can supply some of the answers and, where the answers are still lacking, the tools with which to obtain them. The transition from “simple” statistical mechanics to the more sophisticated versions is undertaken gradually, starting from Chapter 4 to the concluding chapters of the book. The requisite mathematical tools are supplied as needed within each self-contained chapter.

The book was based on the needs of physics graduate students but it is designed to be accessible to engineers, chemists and mathematicians with minimal backgrounds in physics. Too often physics is taught as an idealized science, devoid of statistical uncertainties. An elementary course in thermodynamics and statistical physics can remedy this; Chapters 1–4 are especially suitable for undergraduates aspiring to be theoreticians. Much of the material covered in this book is suitable for self-study but *all* of it can be used as a classroom text in a one-semester course.

Based in part on lecture notes that the author developed during a decade of teaching this material, the present volume seeks to cover many essential physical concepts and theoretical “tricks” as they have evolved over the past two centuries. Some theories are just mentioned while others are developed in great depth, the sole criterion being the author’s somewhat arbitrary opinion of the intellectual depth of the posed problem and of the elegance of its resolution. Here, function follows form.

Specifically, Chapters 1 and 2 develop the rudiments of a statistical science, touching upon metastable states, phase transitions, critical

exponents and the like. Applications to magnetism and superconductivity are included *ab initio*. Chapter 3 recapitulates thermodynamics in a form that invites comparison with the postulational statistical mechanics of Chapter 4. van der Waals gas is studied and then compared to the exactly solved Tonks' gas. Chapters 5 and 6 deal, respectively, with the quantum statistics of bosons and fermions and their various applications. We distinguish the two principal types of bosons: conserved or not. The notion of "quasiparticles" in fermion systems is stressed. We touch upon semiconductor physics and the rôle of the chemical potential $\mu(T)$ in n -type semiconductors, analyzing the case when ionized donors are incapable of binding more than one excess electron due to 2-body forces. Chapter 7 presents the kinetic theory of dilute gases. Boltzmann's H -function is used to compute the approach to thermodynamic equilibrium and his eponymous equation is transformed into an eigenvalue problem in order to solve for the dispersion and decay of sound waves in gases.

Chapter 8 develops the concept of the transfer matrix, including an Onsager-type solution to the two-dimensional Ising model. Exact formulas are used to calculate the critical exponents of selected second-order phase transitions. The concept of "frustration" is introduced and the transfer matrix of the "fully frustrated" two-dimensional Ising model is diagonalized explicitly. A simplified model of fracture, the "zipper", is introduced and partly solved; in the process of studying this "classical" system, we learn something new about the equations of continuity in quantum mechanics!

Chapter 9, the last, is devoted to more advanced techniques: Doi's field-theoretic approach to diffusion-limited chemical reactions is one and the Green's functions theory of the many-body problem is another. As illustrations we work out the eigenvalue spectrum of several special models — including that of a perfectly random Hamiltonian.

Additional models and calculations have been relegated to the numerous problems scattered throughout the text, where you, the reader, can test your mastery of the material. But despite coverage of a wealth of topics this book remains incomplete, as any text of normal length and scope must be. It should be supplemented by the monographs and review articles on critical phenomena, series expansions, reaction rates, exact methods, granular materials, etc., found on the shelves of even the most modest physics libraries. If used to good advantage, the present book could be a gateway to these storehouses of knowledge and research.