CLASSICAL AND COMPUTATIONAL SOLID MECHANICS

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DEDICATION

To my teacher Professor Fung and to Mrs. Fung who have been the constant source of caring, inspiration and guidance; to my parents who were the source of love and discipline; to my wife who has been the source of love, companionship and encouragement; to my children and grandchildren who have been the source of joy and love; to Denny who will be forever in my heart.

Pin Tong

To Luna, Conrad, and Brenda

Y. C.

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PREFACE

The objective of this book is to offer students of science and engineering a concise, general, and easy-to-understand account of some of the most important concepts and methods of classical and computational solid mechanics. The classical part is mainly a re-issue of Fung's *Foundations of Solid Mechanics*, with a major addition to the modern theories of plasticity, and a major revision of the theory of large elastic deformation with finite strains. The computational part consists of five new chapters, which focus on numerical methods to solve many major linear and nonlinear boundary-value problems of solid mechanics.

We hold the principle of easy-to-understand for the readers as an objective of our presentation. We believe that to be easily understood, the presentation must be precise, the definitions and hypotheses must be clear, the arguments must be concise and with sufficient details, and the conclusion has to be drawn very carefully. We strive to pay strict attention to these requirements. We believe that the method must be general and the notations should be unified. Hence, we presented the tensor analysis in general coordinates, but kept the indicial notations for tensors in the first fifteen chapters. In Chapters 16–21, however, the dyadic notations of tensors and the notations for matrix operations are used to shorten the formulas.

This book was written for engineers who invent and design things for human kind and want to use solid mechanics to help implement their designs and applications. It was written also for engineering scientists who enjoy solid mechanics as a discipline and would like to help develop and advance the subject further. It was further designed to serve those physical and natural scientists and biologists and bioengineers whose activities might be helped by the classical and computational solid mechanics. For example, biologists are discovering that the functional behavior of cells depends on the stresses acting on the cell. It is widely recognized that the molecular mechanics of the cell must be developed as soon as possible.

Solid mechanics deals with deformation and motion of “solids.” The displacement that connects the instantaneous position of a particle to its

position in an “original” state is of general interest. The preoccupation about particle displacements distinguishes solid mechanics from that of fluids.

This book begins with an introductory chapter containing a brief sketch of history, an outline of some prototypes of theories, and a description of some more complex features of solid mechanics. In Chapter 2, an introduction to tensor analysis is given. The bulk of the text from Chapters 3 to 16 is concerned with the classical theory of elasticity, but the discussion also includes thermodynamics of solid, thermoelasticity, viscoelasticity, plasticity, and finite deformation theory. Fluid mechanics is excluded, but methods that are common to both fluid and solid mechanics are emphasized. Both dynamics and statics are treated; the concepts of wave propagation are introduced at an early stage. Variational calculus is emphasized, since it provides a unified point of view and is useful in formulating approximate theories and computational methods. The large deflection theory of plates presented in the concluding section of Chapter 16 illustrates the elegance of the general approach to the large deformation theory.

Chapters 17 to 21 are devoted to computational solid mechanics to deal with linear, nonlinear, and nonhomogeneous problems. It is recognized that the incremental approach is the most practical approach, hence the updated Lagrangian description. Chapter 17 develops the incremental theory in considerable detail. Chapter 18 is devoted to numerical methods, with the finite element methods singled out for a detailed discussion of the theory of elasticity. Chapter 19 presents methods of calculations based on the mixed and hybrid variational principles, illustrating the broadening of the computational power with a less restrictive (or weaker) hypothesis in formulating the variational principle. Chapter 20 deals with finite element methods for plates and shells, making the computational methods accessible to the analysis of the structures of aircraft, marine architecture, land vehicles, and shell-like structures in human being, animals, plants, earth, globe, and space. Finally, the book concludes with Chapter 21 dealing with finite element modeling of nonlinear elasticity, viscoelasticity, plasticity, viscoplasticity, and creep. Thus, a broad sweep of modern, advanced topics are covered.

Overall, this book lays emphasis on general methodology. It prepares the students to tackle new problems. However, as it was said in the original preface of the *Foundations of Solid Mechanics*, no single path can embrace the broad field of mechanics. As in mountain climbing, some routes are safe to travel, others more perilous; some may lead to the summit, others to different vistas of interest; some have popular claims, others are less

traveled. In choosing a particular path for a tour through the field, one is influenced by the curricular, the trends in literature, and the interest in engineering and science. Here, a particular way has been chosen to view some of the most beautiful vistas in classical and computational mechanics. In making this choice, we have aimed at straightforwardness and interest, and practical usefulness in the long run.

Holding the book to a reasonable length did not permit inclusion of many numerical examples, which have to be supplemented through problems and references. Fortunately, there are many excellent references to meet this demand. We have presented an extensive bibliography in this book, but we suggest that the reader consults the review journal *Applied Mechanics Reviews* (AMR) published by the American Society of Mechanical Engineers International since 1947, for the current information. The reader is referred to the periodic in-depth reviews of the literature in specific issues of AMR.

We are indebted to many authors and colleagues as acknowledged in the preface of the *Foundations of Solid Mechanics*. In the preparation of the present edition, we are especially indebted to Professors Satya Atluri and Theodore Pian. We would like to record our gratitude to many colleagues who wrote us to discuss various points and sent us errata in the *Foundations of Solid Mechanics*, especially to Drs. Pao-Show D. Cheng, Shun Cheng, Ellis H. Dill, Clive L. Dym, J. B. Haddow, Manohar P. Kamat, Hans Krumhaar, T. D. Leko, Howard A. Magrath, Sumio Murakami, Theodore Pian, R. S. Rivlin, William P. Rodden, Bertil Storakers, Howard J. White, Jr. and John C. Yao. We would also like to thank professors Y. Ohashi, S. Murakami and N. Kamiya for translating the *Foundations* book into Japanese, Professors Oyuang Zhang, Ma Wen-Hua and Wang Kai-Fu for translating the *Foundations* book into Chinese.

On the cover of this book, portraits of some pioneers of mechanics are presented. They are arranged, from left to right and top to bottom, according to their birthdays. They are: Galileo Galilei, 1564-1642, Isaac Newton, 1642-1727, Daniel Bernoulli, 1700-1782, Leonhard Euler, 1707-1783, Joseph Louis Lagrange, 1736-1813, Claude Louis Marie Henri Navier, 1785-1836, Augustin Louis Cauchy, 1789-1857, Lord Kelvin, William Thompson, 1824-1907, Gustave Robert Kirchhoff, 1824-1889, Lord Rayleigh, John William Strutt, 1842-1919, Ludwig Edward Boltzmann, 1844-1906, August Edward Hugh Love, 1863-1940, Stephen P. Timoshenko, 1878-1972. These portraits were supplied by Dr Stephen Juhasz, who was the editor of the *Applied Mechanics Reviews* for 30 years, from 1953-1983. In 1973, Dr Juhasz published an article entitled "Famous Mechanics

Scientists" in *App. Mech. Rev.* Vol. 26, No. 2. These portraits are from that article, in which acknowledgement to original sources is recorded. We thank you, Dr Juhasz.

We would like to take this opportunity to mention a few of editorial notes:

- (1) The bibliography is given at the end of the book.
- (2) Equations in each Section are numbered sequentially. When referring to equations in other Sections, we use the format (Sec. no: Eq. no), e.g. (5.6:4).
- (3) Formulas are concise way of saying lots of things. The most important formulas are marked with a triangular star, $\blacktriangle$. They are worthy of being committed to memory.

Y. C. Fung
P. Tong

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