Hamiltonian Dynamics

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Preface

There are many books on classical mechanics. They can be roughly divided into two classes. One contains books which, in order to be more accessible, are sometimes less transparent with respect to the underlying theoretical structures; the other contains books giving the general, analytical and geometrical, structures of classical mechanics. In the latter, due to greater complexity of the mathematical tools involved, it is however difficult to find books suitable for teaching the subject to graduate students, often because they do not contain a teaching proposal but rather they appear to be written by authors for their colleagues. This book is intended to belong to the second class, but without the shortcoming that was just mentioned.

Part I, Part II and, partially, Part III are intended to be a teaching proposal suitable for graduate students. Thus, they are written from the point of view of a student but with the aim of giving a general understanding of the theory.

Part IV, instead, is concerned with the current research topic of completely integrable field theories and could be even used independently of the others. This part is not written with the same pedagogic spirit that animates the previous chapters and probably it would have required additional chapters concerning the Lagrangian and the Hamiltonian formulation of field theory. However, a pedagogic treatment of the last subject would have taken too much space–time.

I am grateful to my friends:

Giuseppe Marmo, for the invaluable help in reading the manuscript, criticism and important suggestions and for the very many years of common efforts toward an understanding of complete integrability in field theory.

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Finally, I wish to thank Roberto De Luca whose expertise, both in Physics and English, allows me to offer a readable final version. Of course, I am the only one responsible of remaining mistakes.

G. Vilasi

Salerno, April 1998

Introduction

A large amount of scientific activity has been devoted to the asymptotic and geometrical analysis of dynamical systems. This interest was born towards the close of the nineteenth century after the publication of *Les Méthodes Nouvelles de la Mécanique Céleste* in 1892, by Henri Poincaré. The proposed new methods insist on interpreting differential equations as integral curves of vector fields on manifolds, and to analyze the problems concerning long term stability of a dynamical system, for instance, the solar system, by studying their topological properties.

Henri Poincaré was the first to recognize the extraordinarily complicated behavior (today known as *chaos*) of orbits in the vicinity of a separatrix, whose analysis needed the introduction of entirely new mathematics.

Poincaré's suggestion lies in the origin of modern topology, with its powerful tools consisting of tangent and cotangent bundles, differential forms, exterior algebra and calculus, homology and cohomology. All such notions are usually associated with general relativity, string theories, or gauge theories, and not with their main source, *Classical Mechanics*.

On the other hand, in the last few decades there has been a renewed interest in completely integrable Hamiltonian systems, whose concept goes back to the last century, and which, loosely speaking, are dynamical systems admitting a Hamiltonian description and possessing sufficiently many constants of motion, so that they can be integrated by quadratures.

This interest, which previously had considerably weakened, for the really exiguous number of physically prominent examples of completely integrable dynamics with finitely many degrees of freedom, revived with the discovery of

Lax Representation and Inverse Scattering Method. The Lax Representation made possible the solution of many problems of remarkable physical interest as the ones described, for instance, by *Korteweg-de Vries*, *sine-Gordon*, *nonlinear Schrödinger* equations and *Toda's lattice*. All such dynamics are Hamiltonian dynamics on infinite dimensional *weakly-symplectic* manifolds, on which the classical Liouville criterion of integrability can be extended in terms of mixed tensor fields with vanishing Nijenhuis torsion.

Peculiar, in the approach to integrability in terms of invariant mixed tensor fields, is the direct construction of abelian maximal algebras of symmetries, leaving out the associated groups, so that only the algebraic aspect of the traditional methodology is reproduced. An exemplary case is given by the Kepler dynamics, in which both the integrability and the degeneration, classical and quantum, are inferred by identifying the corresponding invariance groups ($SO(3)$ and $SO(4)$).

On the other hand, it is just by means of the modern theory of Hamiltonian systems, based on the analysis of symmetries, that an algebraic group approach arises from the analysis of Lax dynamics. This approach arises from the observation that Hamiltonian dynamics, on the orbits of the coadjoint representation of a Lie group endowed with their natural symplectic structure, are Lax dynamics, provided that an internal product, invariant under the adjoint action, exists in the Lie algebra. The group approach analysis, even if from one side has the merit to be constructive, on the other, is not fit to investigate, *a priori*, the possible integrability of a given dynamics.

In these lectures we shall look at this geometric approach to the study of Hamiltonian dynamical systems, specially in connection with the kinds of problems which arise in completely integrable 2-dimensional field theories.

It would have been interesting to include a chapter concerning nonintegrable dynamics, an essential topic for the theory of particles dynamics in accelerators.

However, this last one is a vast subject and goes beyond the purposes of this book. We will spend however a few words to delineate the idea of invariant tori in phase space, to define and illustrate the structures for organizing dynamics and the origin of chaotic orbits in nonintegrable systems.

Finally, I also hope to lessen the impression, sometimes due to a formal approach, that classical mechanics is a closed subject with no mysteries left to explore.

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