OPTIMAL PORTFOLIOS

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Stochastic Models for Optimal Investment and
Risk Management in Continuous Time

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PREFACE

Nearly 25 years ago, Fischer Black and Myron Scholes came up with their famous option pricing formula which is regarded today as the cornerstone of mathematical finance. However, why should a formula for option pricing be the starting point for writing a book on continuous-time portfolio optimisation? The reason for this lies in the totally different success in the application and science of the Black-Scholes formula and the continuous-time portfolio methods. While both rely on the same type of underlying continuous-time model, the Black-Scholes formula is today common knowledge among practitioners, financial economists and financial mathematicians; everyone at the stock exchange is familiar with it, but only very few people are actually applying continuous-time portfolio methods. For most practitioners and also in large parts of academic circles, the simple one-period mean-variance approach of Markowitz is still state of the art! But this approach was developed 45 years ago! To show that something has really happened in portfolio optimisation theory during the last 45 years is one of the main intentions of this book. Especially, we will concentrate on a continuous-time framework which is the same as the one underlying the Black-Scholes formula.

The whole idea underlying portfolio optimisation is totally natural. One has got a certain amount of money and tries to use it in such a way that one can draw the maximum possible utility from the results of the corresponding activities. This principle covers nearly every situation of daily life. After a hard day's work, you are hungry, thirsty and bored. You quickly walk over to your news agent, choose something to drink, some snacks and a magazine, but a quick look into your wallet tells you that either one drink, one chocolate bar or the magazine is too much. You are facing a problem of portfolio optimisation! Imagine that you are thinking of buying a house and are offered two different ones you can afford. One close to your office with public transport connections, but without a garden and close to a crowded motor way, the other one with a beautiful landscape but requiring you to commute a long distance to work everyday. The decision about which is more convenient for you is in principle (you have definitely guessed it!) a portfolio problem. There are a lot of further examples which you will face without even thinking of applications in finance.

This book should give you an introduction to modern, continuous-time portfolio optimisation starting with the basic (discrete-time) Markowitz mean-variance approach, covering Merton's pioneering work of the late sixties and early seventies, presenting the martingale method as alternative to Merton's method of stochastic

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SOME GUIDELINES AND GENERAL NOTATIONS

There are different ways to read (parts of) this book. The way which is most suitable for you depends on your interests and your knowledge in finance, stochastics, differential equations and optimisation. As continuous-time portfolio models require knowledge on stochastic processes, Itô-calculus and optimisation, the necessary results are provided in the Appendix. As on one hand, it is not possible to give a detailed introduction in these different fields in one single book and on the other hand not necessary to know all the proofs of the relevant results, the Appendix can also serve as an introduction to the above mentioned areas. It is therefore regarded as an essential part of the book which will also open non-experts in stochastic calculus and stochastic control the gate to continuous-time portfolio optimisation. This role of the Appendix is the main reason why it is written in a rather narrative form. However, to allow for a quick way to the central parts of portfolio optimisation, I have decided to leave it in the back of the book and you can judge it on your own if you need to recall some of the basic facts before reading the main subject.

It is definitely recommended to read Chapter 2 (description of the continuous-time model) and at least one of the descriptions of the main approaches to the continuous-time portfolio problem, the stochastic control approach or the martingale method, in Chapter 3. One can then jump to various parts of the book. Only Section 4.4/5 will sometimes require knowledge of the martingale method. Chapter 4 on constrained portfolios is not necessary to understand Chapter 5 on portfolio problems under transaction costs and Chapter 6 on alternative models for portfolio selection.

At some points, it was necessary to use some results which are not given in the index. There, the additional introduction of technicalities and further mathematical theory would have taken much more pages than it could be justified by their actual application. As a compromise, the arguments behind the relevant proofs are sketched and the interested reader is referred to the original sources.

We have used the following hierarchical system for numbering and referring to equations, theorems, remarks and definitions: In each chapter, important equations are numbered consecutively starting with (1). By referring to such an equation given in the same chapter, we only use this bracketed number. If it is referred to an equation occurring in a different chapter then we add the chapter number inside the bracket, i.e. equation (3.12) is equation (12) of Chapter 3. Theorems, remarks, lemmas, definitions, ... are all put together in one group with respect to numbering.

As with the equations, they are numbered consecutively in each chapter and are referred to by their number without chapter number if they are in the same chapter or referred with chapter number and number if they are from a different chapter. For example, Remark 4.5 has the “group number” 5 in Chapter. Chapters are decomposed into sections (numbered by Arabian digits) which are — sometimes — further split up into sub-sections (numbered by small Latin numbers). The Appendix is made up of five different parts, A – E, each of them having sub-parts which are numbered consecutively. Also, equations, theorems, remarks,... are numbered as in the main chapters. Results from the Appendix are always referred to in the book by putting the letter of the relevant part of the Appendix in front of their number.

Most of the notation used in the book is explained where it is used first. However, we will explain some frequently used notation and abbreviations below.

Abbreviations

iid	independent, identically distributed
ode	ordinary differential equation
pde	partial differential equation
qvi	quasi-variational inequality
sde	stochastic differential equation
w.l.o.g.	without loss of generality
s.t.	such that

Some sets and spaces

$\mathbf{N}$	$:= \{1, 2, 3, \dots\}$
$\mathbf{R}^d$	d -dimensional space of real numbers
$C^{i,j}(A, B)$	space of functions $f: A \times B \rightarrow \mathbf{R}^d$ which are i times continuously differentiable with respect to the variables corresponding to A and j times continuously differentiable with respect to the variables corresponding to B
$\sigma(A)$	the smallest σ -algebra containing the collection of sets A
$F \otimes G$	the product σ -algebra of the two σ -algebras F and G

Notations

x'	denotes the transposed of the vector x
$\ x\ $	$:= \sqrt{\sum_{i=1}^n x_i^2}$, i.e. the Euclidean norm of $x \in \mathbf{R}^n$
$a \wedge b$	$:= \min\{a, b\}$, i.e. the minimum of the real numbers a and b
x^+	$:= \max\{0, x\}$, i.e. the positive part of $x \in \mathbf{R}$
x^-	$:= \max\{0, -x\}$, i.e. the negative part of $x \in \mathbf{R}$
L	Lebesgue measure

$L \otimes P$	product measure of the two measures L and P
$\underline{1}$	$:= (1, \dots, 1)'$ (of appropriate dimension)
e_i	$:= (0, \dots, 0, 1, 0, \dots, 0)'$ the i th unit vector (i.e. the one having zero entries for all components but number i where it has “1” as entry)
I	the identity matrix of appropriate dimension
$1_A(x)$	$:= \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$, i.e. the indicator function of the set A
$\max_x f(x)$	maximise the function $f(x)$ with respect to x
$\max_x f(x)$	maximise the function $f(x)$ over all $x \in A$
s.t. $x \in A$	