Asymptotic Models of Fields in Dilute and Densely Packed Composites

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Preface

This book is about asymptotic models for problems of elasticity, electrostatics and electromagnetism describing physical phenomena in heterogeneous composite structures. Particular attention is paid to analysis of structures containing inclusions or voids which are either of small relative volume (dilute composites) or are placed close to each other (densely packed composites).

The methods described in this text are analytical, and the range of our interests covers two areas: (a) the method of compound asymptotic expansions applied to singularly perturbed boundary value problems and (b) the multipole method which proves to be efficient in analysis of fields for domains containing arrays of inclusions of circular or spherical shapes.

The book came as a result of our recent work on mathematical modelling of defects in electromagnetism and elasticity. One simple and efficient method for the study of small defects is via evaluation of their dipole tensors and the corresponding energy change associated with the perturbation field. However, when inclusions are finite in size and interact with each other one needs the high-order multipole approximations of solutions.

A particular feature of singularly perturbed problems is the presence of so-called boundary layer fields concentrated in the high-gradient regions. Boundary layers are usually described by solutions of model problems posed in unbounded domains. In some cases one can obtain these solutions explicitly or evaluate their asymptotics at infinity. In this text we study models of solids containing small inclusions or voids, and the boundary layers describe perturbations of elastic fields associated with these inclusions. It is shown that the leading asymptotic representation of a boundary layer at

infinity is determined by components of a dipole tensor of the inclusion, and by the remote load applied on the exterior boundary of the domain. The analytical technique we use to model small inclusions is known in the literature as the method of compound asymptotic expansions. The theory of compound asymptotic expansions was created during the last two decades, and the key results in this development to date belong to Maz'ya *et al.* (2001) who have written a two-volume monograph, which is the most comprehensive text on the theory of singularly perturbed elliptic boundary value problems at present.

The dipole fields associated with inhomogeneities in electrostatics, electromagnetism, fluid mechanics and elasticity were used in many applications to evaluate the energy of the perturbation fields as well as to determine effective moduli of composites with periodic structures (see for example the classical work of Lord Rayleigh (1982), Pólya and Szëgo (1951) and G.I. Taylor (1928)). A systematic analytical outline of properties of dipole tensors for a class of boundary value problems for the Laplacian was included in a book on isoperimetric inequalities in mathematical physics by Pólya and Szëgo (1951). Further studies of dipole tensors in vector problems of elasticity were published by Movchan and Serkov (1997). The book by Movchan and Movchan (1995) outlines applications of dipole tensors in certain classes of asymptotic models of fracture mechanics.

A new "shield effect" was discovered by Valentini, Serkov, Bigoni and Movchan (1999) for coated inclusions in elastic media. It has been shown that one can choose parameters of an elastic coating in such a way that an inclusion becomes "neutral": if placed in a constant stress field the inclusion does not produce any elastic energy change. For scalar problems associated with the Laplace operator, structures of this kind have been described in the books by Cherkaev (2000) and Milton (2002).

Dipole tensors were also efficiently used in the asymptotic analysis of a class of eigenvalue problems. In the papers by Movchan (1988, 2001) one can find asymptotic algorithms for models of vibration of domains containing small inclusions. The paper by Movchan and Nazarov (1990) and recent publications by Esparza and Movchan (1998), and Esparza (2002) contain asymptotic studies of singularity exponents at the vertices of conical defects and cones with imperfect bonding over their lateral surface.

For inclusions of circular or spherical shapes, the perturbation fields can be constructed explicitly even for the case when a body contains an array of these defects placed close to each other. Inevitably, it involves analysis of

interaction between different inclusions within the array and requires high-order multipole representations of solutions. The multipole method for heterogeneous domains was introduced over a hundred years ago by Lord Rayleigh (1892) who studied transport properties of conducting media containing periodic arrays of circular or spherical dielectric inclusions. This study was extensively developed and applied to models of two-phase composite structures in elasticity and electro-magnetism. McPhedran, Milton and Poladian (1988) introduced an asymptotic approximation for multipole coefficients in the representation of an electrostatic potential around circular dielectric inclusions, which are close to touching. This analysis was extended further to vector problems of two-dimensional elasticity by McPhedran and Movchan (1994). A new original development for spectral problems of electromagnetism has been published in a series of papers by McPhedran and his colleagues (1982, 1994, 1995, 1996, 1997) who adopted the multipole method to analysis of dispersion diagrams for photonic band gap composite structures used in the design of photonic crystal fibres.

The technological and theoretical motivations are linked to the design of modern fibre-optics communication lines and optical filters. One of the most advanced recent studies of photonic crystal fibres that combines practical implementation of band-gap structures with analytical and numerical modelling was presented by Liu, Russell and Dong (1998), Mogilevtsev, Birks and Russell (1999), Russell and Liu (2000) and Diez *et al.* (2000).

In the paper by Poulton *et al.* (2000) a generalisation of the original multipole method was developed to analyse propagation of elastic waves through a two-dimensional doubly periodic array of circular inclusions. An homogenised elastic composite material is, in general, anisotropic in the long-wave approximation. Such a material may also exhibit interesting filtering properties for the case when the wavelengths are comparable with the scale size of the periodic structure. In this book we shall show some asymptotic features of multipole solutions for the cases of small inclusions (dilute composites), inclusions close to touching (dense packing) and high-contrast inclusions.

The plan of this book is as follows.

We begin with a simple introduction where we talk about the compound asymptotic expansions technique applied to boundary value problems posed in domains containing small inclusions. Dipole tensors are defined both for scalar boundary value problems for the Laplacian and for vector problems of elasticity. Constructive methods are presented for evaluation of dipole

tensors, and examples of defects of “equivalent shapes” are discussed in detail. Further, for the case of voids close to touching we introduce an asymptotic algorithm based on a multipole method for circular inclusions. The reader can see a link between Chapter 1 and the final Chapter 3, where the multipole methods are described in detail. A correspondence is also established between arrays of voids close to touching and lattice structures that exhibit filtering properties for waves of certain frequencies.

In Chapter 2 we show how dipole tensors can be used in spectral problems involving domains with small defects. The main study is allocated for singularity exponents at the vertices of thin conical inclusions. Examples of “imperfect interfaces” considered in this chapter include the cases of thin and soft elastic coatings.

Chapter 3 describes a multipole method, originally due to Lord Rayleigh (1892). The first two sections of this chapter deal with static problems (both electrostatics and elasticity) in composite structures containing doubly periodic arrays of circular inclusions. We also study the asymptotic problems involving dilute densely packed structures and two-phase high-contrast composites. Finally we present a version of the multipole method for eigenvalue problems of electromagnetism and elasticity, and our main aim is to discuss the structure of dispersion diagrams associated with electromagnetic and elastic waves propagating across the composite. Constructive algorithms are also given for evaluation of values of the effective refractive index for doubly periodic composites.

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