PSEUDO DIFFERENTIAL OPERATORS & MARKOV PROCESSES

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PSEUDO DIFFERENTIAL OPERATORS & MARKOV PROCESSES

Volume II

Generators and Their Potential Theory

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ICP

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PSEUDO-DIFFERENTIAL OPERATORS AND MARKOV PROCESSES

Volume II: Generators and Their Potential Theory

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Contents

Preface	vii
Notation	ix
General Notation	ix
Functions and Distributions	xi
Measures and Integrals	xi
Spaces of Functions, Measures and Distributions	xii
Some Families of Functions	xiv
Norms, Scalar Products and Seminorms	xiv
Notation from Functional Analysis, Operators	xv
Notations related to Pseudo-Differential Operators	xvi
Notations related to Potential Theory and Harmonic Analysis	xvi
Introduction: Pseudo-Differential Operators and Markov Processes	xix
II Generators and Their Potential Theory	1
1 Introduction	3
2 Generators of Feller and Sub-Markovian Semigroups	13
2.1 Second Order Elliptic Differential Operators as Generators of Feller and Sub-Markovian Semigroups	13
2.2 Some Second Order Differential Operators with Non-Negative Characteristic Form as Generators of Sub-Markovian Semigroups	48
2.3 Some Properties of Pseudo-Differential Operators with Negative Definite Symbols	66

2.4	Hoh's Symbolic Calculus for Pseudo-Differential Operators with Negative Definite Symbols	87
2.5	Estimates for Pseudo-Differential Operators with Negative Definite Symbols Using the Symbolic Calculus	113
2.6	Feller Semigroups and Sub-Markovian Semigroups Generated by Pseudo-Differential Operators	126
2.7	Further Analytic Approaches for Constructing Feller and Sub-Markovian Semigroups	138
2.8	Some Perturbation Results	151
2.9	On Semigroups Obtained by Subordination	172
2.10	Pseudo-Differential Operators with Variable Order of Differentiation as Generators of Feller Semigroups	204
3	Potential Theory of Semigroups and Generators	213
3.1	Capacities and Abstract Bessel Potential Spaces	213
3.2	First Results on L^p -Sub-Markovian Semigroups in their Associated Bessel Potential Spaces	244
3.3	Bessel Potential Spaces Associated with a Continuous Negative Definite Function	270
3.4	Stein's Littlewood-Paley Theory for Sub-Markovian Semigroups	300
3.5	Global Properties of L^p -Sub-Markovian Semigroups	336
3.6	Nash-Type and Sobolev-Type Inequalities — a Short Outline .	409
	Bibliography	417
	Author Index	443
	Subject Index	447

Preface

This is the second volume of my treatise on “Pseudo-differential operators and Markov processes”, and it covers two topics: Generating Feller and sub-Markovian semigroups by starting with pseudo-differential operators with negative definite symbols, and the potential theory, more correctly parts of the potential theory, of these operators and their corresponding semigroups.

The style of the presentation changes in so far as we address the book to readers being familiar with the essential content of volume I.

Chapter 2 relies essentially on work done by my former PhD-students and myself and this chapter includes many results being presented for the first time in a monograph. I owe special thanks to W.Hoh and R.L.Schilling for allowing me to incorporate their work and ideas, partly following closely their presentations. This chapter provides large classes of pseudo-differential operators generating Feller and sub-Markovian semigroups — in a certain sense it shows the richness of the theory.

Chapter 3 is devoted to the potential theory of semigroups generated by pseudo-differential operators with negative definite symbols. In principle we provided non-trivial, new examples in Chapter 2 for various aspects of potential theory. Therefore it would be straightforward just to work out these examples. However, in order to obtain a more self-contained presentation we decided to include larger parts of “standard” potential theory. But we do not believe that we have to “re-invent the wheel”, i.e. in the general theory we followed partly rather closely existing work, stressing however these aspects which are of special interest and need in handling our examples. In addition we had to extend the theory and added quite a lot of new material. Again I owe thanks to my collaborators W.Farkas, W.Hoh and R.L.Schilling for allowing me to include their (or our) work. But even more, I owe thanks to those authors from whom I borrowed for my presentation. In the “Introduction” and often in the text we make rather precise references, in fact we preferred to have the

references close to the results and thus we need no additional notes.

No doubt, the following passage taken from the preface of volume I remains valid:

“One of the major difficulties in writing the book was that in different fields of mathematics different notations and normalizations were used. . . . A second, much larger problem was and still is the enormous amount of work done in fields we have to deal with.”

Hence it is clear that we could not cover all material related to our topic and we hope that we succeeded to give fair credits.

Preparing this volume I received support from many sides. First of all I want to thank R.L.Schilling who read the whole manuscript, suggested once again many improvements and partly, proposed new or different proofs, or even proved new results which I was allowed to include immediately. Secondly, I want to thank W.Hoh for comments to some earlier draft of parts of the manuscript, but in particular, to allow me to use for writing Chapter 2 to larger extent his “Habilitationsschrift” which unfortunately was never published separately.

Special thanks go to M.Röckner and H.Triebel for support in a critical academic year 1999/2000, as well as to my department at the University of Wales Swansea, especially A.Truman, where the major part of the book was written in the last one and a half year.

Many thanks to V.Knopova for transforming my handwritten manuscript into a $\text{\TeX}$ -file, and to B.Boettcher for proof-reading the type-written version. A.G.Tokarev is to thank for putting the $\text{\TeX}$ -file into its final form including doing all corrections, and I.M.Davies helped whenever the computer system made some difficulties — many thanks to all of them.

During the work on the book I received some financial support from DFG, contracts Ja 552_{7-1,7-2}, and EPSRC, contract GR/R40241/01.

Last but not least many thanks to Ye Qiang and his team in Singapore who made the cooperation with the publisher a pleasure.

Swansea, April 2002

Niels Jacob

Notation

I.*xyz* means: page *xyz* in volume I

General Notation

$\mathcal{P}(G)$ power set of G

$\mathbb{N}$ natural numbers

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$\mathbb{N}_0^n$ set of all multiindices

$\mathbb{N}_{0,\gamma}^n = \{\beta \in \mathbb{N}_0^n; |\beta| > 0 \text{ and } 0 < |\beta| \leq \gamma\}$

$\mathbb{Z}$ integers

$\mathbb{Q}$ rational numbers

$\mathbb{R}$ real numbers

$\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$

$\mathbb{R}^n$ euclidean vector space

$Q_R = \{x \in \mathbb{R}^n; |x_i| \leq R\}$

$B_R(a) = \{x \in \mathbb{R}^n; |x - a| < R\}$

$S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$

ω_n volume of $B_1(0) \subset \mathbb{R}^n$

$T_x \mathbb{R}^n$ tangent space to $\mathbb{R}^n$ at x

$\mathbb{C}$ complex numbers

$S_{\theta,d_0} = \{z \in \mathbb{C}; |\arg(z - d_0)| < \theta\}$

$\mathbb{C}^n$ unitary vector space

$a \wedge b = \min(a, b)$

$a \vee b = \max(a, b)$

$\text{sign } a = \begin{cases} \frac{a}{|a|}, & a \neq 0 \\ 0, & a = 0 \end{cases} \quad \text{signum of } a$

- $|A|$ cardinality of a finite set
 $A \setminus B$ set theoretical difference of two sets
 A^c complement of the set A
 $\text{diag } A = \{(x, x); x \in A\}$
 $A + B = \{x + y; x \in A \text{ and } y \in B\}$
 $\lambda A = \{\lambda x; x \in A\}, \lambda \in \mathbb{R} \text{ or } \lambda \in \mathbb{C}$
 $\text{lin } A$ linear hull of A
 $\text{conv } A$ convex hull of A
 $\text{ext } A$ extremal points of A
 $H_1 \oplus H_2$ direct sum in a vector space
 $\bar{A}$ closure of a set
 $\overset{\circ}{A}$ (open) interior of a set
 ∂A boundary of a set
 $\partial_H K$ Choquet boundary
 $A_\infty = A \cup \{\infty\}$ one point compactification of A
 $\mathcal{O}_\infty$ topology of A_∞
 $|\alpha| = \alpha_1 + \dots + \alpha_n$ for $\alpha \in \mathbb{N}_0^n$
 $\alpha! = \alpha_1! \dots \alpha_n!$ for $\alpha \in \mathbb{N}_0^n$
 $\alpha \leq \beta$ means $\alpha_i \leq \beta_i$ for $\alpha, \beta \in \mathbb{N}_0^n$
 $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \dots \binom{\alpha_n}{\beta_n}$ for $\alpha, \beta \in \mathbb{N}_0^n$
 $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$
 $\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$
 $D^\alpha u = (-i\partial)^\alpha u$
 $\frac{d^+}{dt} u(x) = \lim_{t \rightarrow 0^+} \frac{u(x+t) - u(x)}{t}$
 $\frac{d^-}{dt} u(x) = \lim_{t \rightarrow 0^-} \frac{u(x+t) - u(x)}{t}$
 $\bar{\partial}_z u = \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u, z = x + iy$
 $I^{(\alpha)}$ fractional integral
 D_+^α, D_-^α fractional derivative for $\alpha \in \mathbb{R} \setminus \mathbb{N}$
 $\text{ind}_\gamma(z)$ index of $z \in \mathbb{C}$ with respect to a closed path γ
 $j \succeq j_0$ order relation in a net
 $\lim_{j \in I}$ limit with respect to a net

Functions and Distributions

χ_A	characteristic function of the set A
id_A	identity map in A
$f _A$	restriction of f to A
f^+	positive part of f , i.e. $f^+ = f \vee 0$
f^-	negative part of f , i.e. $f^- = -(f \vee 0)$
$\text{Re } f$	real part of a function
$\text{Im } f$	imaginary part of a function
f_∞	function extended from A to A_∞ by $f(\infty) = 0$
$(f_\nu)_{\nu \in \mathbb{N}}$	sequence of functions
$f_\nu \uparrow f$	increasing sequence of functions tending to f
$f_\nu \downarrow f$	decreasing sequence of functions tending to f
$f \circ g$	composition of functions
$f \otimes g$	tensor product of functions (or distributions)
$f * g$	convolution of functions (or distributions)
$j_\varepsilon, J_\varepsilon(u)$	Friedrichs mollifier
$Sx = -x$	reflection at the origin
$T_a x = x + a$	translation by a
$H_\lambda x = \lambda x$	homothetic map
$\pi_j(x) = x_j$	j -th coordinate projection
$B(\alpha, \beta)$	Beta-function
$J_{\frac{n}{2}}$	Bessel function of first kind and order $\frac{n}{2}$
K_1	Bessel function of third kind and index 1
$r_\nu(t)$	Rademacher functions
T^*u	pullback of a function or distribution w.r.t. the diffeomorphism T
$\hat{u}, Fu$	Fourier transform
$F^{-1}u$	inverse Fourier transform
$\left. \begin{array}{l} \delta_{diag} \\ \text{id}_{kernel} \end{array} \right\}$	kernel of $\text{id}_{C_0^\infty(G)}$
$\text{supp } u$	support of a function or distribution
$\text{ess sup } u$	essential supremum of a function
sing supp	singular support of a function or distribution

Measures and Integrals

$\mathcal{B}^{(n)}$	Borel sets in $\mathbb{R}^n$
$\mathcal{B}(G)$	Borel sets of a topological space G

- $\sigma(S)$ σ -field generated by S
 $d(S)$ Dynkin system generated by S
 $\lambda^{(n)}$ Lebesgue measure in $\mathbb{R}^n$
 ε_a Dirac measure at $a \in \mathbb{R}^n$
 ε Dirac measure at $0 \in \mathbb{R}^n$
 $\mu_1 \otimes \mu_2$ product of the measures μ_1 and μ_2
 $\mu_1 * \mu_2$ convolution of the measures μ_1 and μ_2
 $\mu(\varphi) = \int \varphi d\mu$
 $\|\mu\|$ total mass of a measure μ
 $\text{supp } \mu$ support of a measure μ
 $(\mu_t)_{t \geq 0}$ convolution semigroup of subprobabilities
 $(\mu_t^f)_{t \geq 0}$ subordinate convolution semigroup
 $(\sigma_t^\alpha)_{t \geq 0}$ one-sided symmetric stable semigroup
 $v.p.$ principal value of an integral
 $\int_{a+}^b = \int_{(a,b]}$
 $\int_a^{b+} = \int_{[a,b)}$
 $\oint_\gamma$ integral over the closed path γ

Spaces of Functions, Measures and Distributions

If $X(\Omega)$ is one of the spaces in the following list, $X(\Omega, \mathbb{R})$ stands for the subspace of real-valued elements of $X(\Omega)$. In the following $G \subseteq \mathbb{R}^n$ stands always for an open set, $K \subset \mathbb{R}^N$ for a compact set.

- $A(\mathbb{R}^n)$ Wiener algebra
 $B(\Omega)$ Borel measurable function
 $B_b(\Omega)$ bounded Borel measurable function
 $B_{pq}^s(\mathbb{R}^n)$ Besov space
 $B_{k,p}(\mathbb{R}^n)$ Hörmander space, see page I.207
 $B_{\psi,p}^s(\mathbb{R}^n)$ see page I.207
 $B_{\psi_1,\psi_2,p}^s(\mathbb{R}^n)$ see page I.212
 $C(G)$ continuous functions
 $C_0(G)$ continuous functions with compact support
 $C_\infty(G)$ continuous functions vanishing at infinity
 $C_u(G)$ uniformly continuous and bounded functions
 $C_b(G)$ bounded continuous functions
 $C^{0,\lambda}(G)$ Hölder continuous functions with exponent $\lambda \in (0, 1]$

- $C(\bar{G}) = \{u \in C(G); u \text{ extends continuously to } \bar{G}\}$
 $C^m(G)$ m -times continuously differentiable functions
 $C_0^m(G) = C^m(G) \cap C_0(G)$
 $C_\infty^m(G) = \{u \in C^m(G); \partial^\alpha u \in C_\infty(G), |\alpha| \leq m\}$
 $C_u^m(G) = \{u \in C^m(G); \partial^\alpha u \in C_u(G), |\alpha| \leq m\}$
 $C_b^m(G) = \{u \in C^m(G); \partial^\alpha u \in C_b(G), |\alpha| \leq m\}$
 $C_\infty^{m,\lambda}(G) = \{u \in C^m(G); \partial^\alpha u \in C^{0,\lambda}(G), |\alpha| = m\}$
 $C^m(\bar{G}) = \{u \in C^m(G); \partial^\alpha u \in C(\bar{G}), |\alpha| \leq m\}$
 $C^\infty(G) = \cap_{m \in \mathbb{N}} C^m(G)$
 $C_0^\infty(G) = \cap_{m \in \mathbb{N}} C_0^m(G)$
 $C_0^\infty(K)$ functions which are arbitrarily often differentiable and have compact support in K
 $\mathcal{D}'(G)$ space of all distributions
 $\mathcal{E}'(G)$ space of all distribution with compact support
 $F_{pq}^s(\mathbb{R}^n)$ Triebel–Lizorkin space
 $H_2^m(\mathbb{R}^n)$ classical Sobolev space of integer order
 $H^{m,p}(\mathbb{R}^n)$ classical L^p -Sobolev space of integer order
 $H^s(\mathbb{R}^n)$ classical Sobolev space of fractional order $s \in \mathbb{R}$
 $H^{\psi,s}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); \|u\|_{\psi,s} < \infty\}$
 $H_p^{\psi,2}(\mathbb{R}^n)$ see Definition 3.3.3 on page 275
 $H_p^{\psi,s}(\mathbb{R}^n)$ see (3.166) on page 279
 $H_{loc}^{\psi,s}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); \varphi u \in H^{\psi,s}(\mathbb{R}^n) \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^n)\}$
 $l_q = \{(a_k)_{k \in \mathbb{N}_0}; a_k \in \mathbb{C} \text{ and } \|(a_k)\|_{l_q} < \infty\}$
 $L^p(\Omega, \mu)$ usual Lebesgue space over $(\Omega, \mathcal{A}, \mu)$
 $L^p(G) = L^p(G, \lambda^{(n)})$
 $L_{loc}^1(G) = \{u \in B(G); \chi_K u \in L^1(G) \text{ for all } K \subset G \text{ compact}\}$
 $\mathcal{M}^+(\Omega)$ measures on Ω
 $\mathcal{M}_b^+(\Omega)$ bounded measures on Ω
 $\mathcal{M}_b^1(\Omega)$ probability measures on Ω
 $\tilde{\mathcal{M}}_b(X)$ signed Radon measures on a locally compact space X with finite total mass
 $\tilde{\mathcal{M}}^{\mathbb{C}}(\Omega)$ complex measure on Ω
 $\mathcal{M}_b^{\mathbb{C}}(\Omega)$ bounded complex measure on Ω
 $M_{p,q}$ Fourier multipliers of type (p, q)
 $\mathcal{S}(\mathbb{R}^n)$ Schwartz space of tempered functions
 $\mathcal{S}'(\mathbb{R}^n)$ tempered distributions
 $W^{m,p}(\mathbb{R}^n)$ classical L^p -Sobolev space of integer order
 $\mathring{W}^{m,p}(\mathbb{R}^n)$ see (2.23) on page 21

$W^{s,p}(\mathbb{R}^n)$ see (2.417) page 167

Some Families of Functions

$CN(\mathbb{R}^n)$ continuous negative definite functions

$CP(\mathbb{R}^n)$ continuous positive definite functions

$\mathcal{H}_{l.s.c.}(G)$ lower semicontinuous functions

$\mathcal{H}_{u.s.c.}(G)$ upper semicontinuous functions

$\mathcal{H}(A)$ see page I.62

$\mathcal{H}(X_0, X_1)$ see page I.71

$N(\mathbb{R}^n)$ negative definite functions

$P(\mathbb{R}^n)$ positive definite functions

$\mathcal{S}$ Stieltjes functions

$\mathcal{S}_0(\mathbb{R}^n) = \{u \in \mathcal{S}(\mathbb{R}^n); \hat{u}(0) = \frac{\partial \hat{u}}{\partial \xi_i}(0) = 0 \text{ for } 1 \leq j \leq n\}$

Norms, Scalar Products and Seminorms

$|x|$ euclidean distance in $\mathbb{R}^n$

$|x|_\infty = \max\{|x_1|, \dots, |x_n|\}$, $x \in \mathbb{R}^n$

$|x|_q = \left(\sum_{j=1}^n |x_j|^q\right)^{1/q}$

$|z|$ euclidean distance in $\mathbb{C}^n$

$\|\nabla^j u\|_{L^p}$ see (2.5) on page 16

$\|u\|_X$ norm of u in the space X

$\|u\|_{A,X,Y} = \|u\|_X + \|Au\|_Y$
 $\|u\|_{A,X} = \|u\|_X + \|Au\|_X$ } graph norm with respect to the operator A

$\|A\| = \|A\|_{X,Y}$ operator norm of the operator A

$\|u\|_{x_\theta}$ see page I.72

$\|u\|_{\mathcal{H}}$ see page I.71

$\|u\|_0, (u, u)_0$ norm and scalar product in $L^2(\Omega, \mu)$

$\|u\|_\infty = \sup |u(x)|$ or $\text{ess sup } |u(x)|$

$\|u\|_s$ norm in the space $H^s(\mathbb{R}^n)$

$\|u\|_{m,2}$ norm in the space $H_2^m(\mathbb{R}^n)$ and $W^{m,2}(\mathbb{R}^n)$

$\|u\|_{\psi,s}$ norm in the space $H^{\psi,s}(\mathbb{R}^n)$

$\|u\|_{\psi,s,p}$ norm in the space $B_{\psi,p}^s(\mathbb{R}^n)$

$\|u\|_{\psi_1,\psi_2,s,p}$ norm in the space $B_{\psi_1,\psi_2,p}^s(\mathbb{R}^n)$

$\|(a_k)\|_{l_q} = \left(\sum_{k=0}^\infty |a_k|^q\right)^{1/q}$

$\|(f_k)\|_{L^p(l_q)}$ see page I.227

$\|(f_k)\|_{l^q(L_p)}$ see page I.227

$p_{m_1, m_2}(u) = \sup_{x \in \mathbb{R}^n} ((1 + |x|^2)^{m_1/2} \sum_{|\alpha| \leq m_2} |\partial^\alpha u(x)|)$

$p_{\alpha, \beta}(u) = \sup_{x \in \mathbb{R}^n} |x^\beta \partial^\alpha u(x)|$

$p_{\alpha, K}(u) = \sup_{x \in K} |\partial^\alpha u(x)|, \quad u \text{ compact}$

$q_{m, K}(u) = \sum_{|\alpha| \leq m} \sup_{x \in K} |\partial^\alpha u(x)|, \quad K \text{ compact}$

Notation from Functional Analysis, Operators

$\mathcal{B}_0$ base of neighbourhoods of $\mathcal{O} \in X$, X vector space

$\mathcal{B}_X = \{x + B_0, B_0 \in \mathcal{B}_0\}$

$U_{p, \varepsilon} = \{x \in X; p(x) \leq \varepsilon\}$, X vector space, p seminorm

$(X, \|\cdot\|_X)$ Banach space X with norm $\|\cdot\|_X$

X^* dual space of a topological vector space

$X \hookrightarrow Y$ continuous embedding of X into Y

$[X_0, X_1]_\theta$ complex interpolation space

$B(X, Y)$ bounded linear operators from X into Y

$B(X)$ bounded linear operators from X into itself

$s\text{-lim}$ strong limit

$w\text{-lim}$ weak limit

$\langle u, x \rangle$ duality pairing between X^* and X

$(A, D(A))$ linear operator with domain $D(A)$

$D(A)$ domain of an operator

$R(A)$ range of an operator

$\Gamma(A)$ graph of an operator

$\text{Ker}(A)$ kernel (null-space) of an operator

$\bar{A}$ closure of an operator

A^* conjugate operator or adjoint Hilbert space operator

$\rho(A)$ resolvent set of an operator

$\rho_+(A) = \rho(A) \cap (0, \infty)$

$\sigma(A)$ spectrum of an operator

$\theta(A)$ numerical range of an operator

$A_\lambda = \lambda A(\lambda - A)^{-1}$ Yosida approximation of A

$(R_\lambda)_{\lambda > 0}$ resolvent of an operator

$R_\lambda = (\lambda - A)^{-1}$

$B_\lambda(u, v) = B(u, v) + \lambda(u, v)_0$ for a bilinear form B

- B^{sym} symmetric part of a bilinear form B
 B^{asym} antisymmetric part of a bilinear form B
 K_{op} operator with kernel K
 $q(x, D)$ pseudo-differential operator with symbol $q(x, \xi)$
 $\psi(D)$ pseudo-differential operator with symbol $\psi(\xi)$
 $(T_t)_{t \geq 0}$ one parameter semigroup of operators
 $(T_t^*)_{t \geq 0}$ adjoint semigroup to $(T_t)_{t \geq 0}$
 $(T_t^f)_{t \geq 0}$ subordinated semigroup
 A^f generator of subordinated semigroup
 $(T_t^{(\infty)})_{t \geq 0}$ semigroup on $C_\infty(\mathbb{R}^n)$
 $A^{(\infty)}$ generator of $(T_t^{(\infty)})_{t \geq 0}$
 $(T_t^{(p)})_{t \geq 0}$ semigroup on $L^p(\mathbb{R}^n)$, $1 < p < \infty$
 $A^{(p)}$ generator of $(T_t^{(p)})_{t \geq 0}$

Notations related to Pseudo-Differential Operators

- $S_{1,0}^m(\mathbb{R}^n)$ see Example 2.4.8 on page 92
 $S_{\rho,\delta}^m(\mathbb{R}^n)$ see Definition 2.7.1 on page 139
 $CN S_{\rho,\delta}^m(\mathbb{R}^n)$ see Definition 2.7.7 on page 140
 $S_\rho^{m,\psi}(\mathbb{R}^n)$ see Definition 2.4.4 on page 89
 $S_0^{m,\psi}(\mathbb{R}^n)$ see Definition 2.4.4 on page 89
 $S_0^{m,m',\psi}(\mathbb{R}^n)$ see Definition 2.4.16 on page 97
 $\Psi_\rho^{m,\psi}(\mathbb{R}^n)$ see Definition 2.4.10 on page 93
 $\Psi_0^{m,\psi}(\mathbb{R}^n)$ see Definition 2.4.10 on page 93
 Λ see Definition 2.4.3 on page 89
 $\mathcal{A}$ see Definition 2.4.12 on page 94
 $O_S - \int$ oscillatory integral, see Definition 2.4.12 on page 94
 γ^σ see (2.361) on page 147
 $\gamma(dx, dx)$ see page 147

Notations related to Potential Theory and Harmonic Analysis

- $\text{Ana}(G)$ analytic subsets of G
 cap^* outer capacity

cap_* inner capacity

cap capacity

$\text{cap}_{r,p}$ capacity in $\mathcal{F}_{r,p}$

$\text{cap}_{r,p}^\psi$ capacity in $H_p^{\psi,r}(\mathbb{R}^n)$

$V_r^{(p)}$ Γ -transform of $(T_t^{(p)})_{t=0}$, see Definition 3.1.17 on page 224

$\mathcal{F}_{r,p} = V_r^{(p)} L^p$ Bessel potential space of order r associated with $(T_t^{(p)})_{t=0}$

$\mathcal{F}_e$ extended Dirichlet space

$f^*(u)$ see (3.241) on page 303

$g_1(u)$ Littlewood–Paley function of order 1, see (3.294) on page 324

$g_k(u)$ Littlewood–Paley function of order k , see (3.298) on page 325

$u_{(p)}^*(x) = \sup_{t>0} |T_t^{(p)} u(x)|$

$u_{k,(p)}^*(x) = \sup_{t>0} \left| t^k \frac{\partial^k T_t^{(p)} u(x)}{\partial t^k} \right|$

$E(f)$ expectation of f

$E(f|\mathcal{F})$ conditional expectation of f w.r.t. $\mathcal{F}$

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Introduction:

Pseudo-Differential Operators and Markov Processes

This monograph is devoted to Markov processes with state space $\mathbb{R}^n$, in particular we will discuss jump processes. A brief, but very readable description of the development of the theory of stochastic processes until the late fifties can be found in the preface of the monograph of K.Itô and H.McKean [145]. In particular the different ways of constructing processes are pointed out: constructions using (functional) analysis (N.Wiener, A.Kolmogorov, W.Feller, K.Yosida, E.B.Dynkin) and pathwise constructions (P.Lévy, K.Itô).

Today, pathwise constructions via stochastic differential equations, for example, are in the centre of probabilists' interest and they led to the field of stochastic analysis, i.e. analysis on path spaces which is since the publication of P.Malliavin's paper [205] one of the central themes in modern probability theory. The reader is highly recommended to P.Malliavin's essays [207] to get first hand information.

However, after the publication of M.Fukushima's work [97] on Dirichlet forms and Markov processes the (functional) analytic approach to stochastic processes returned also into the centre of probabilists' interest.

In this monograph we will emphasize another point of view, namely the relation of Fourier analysis and Markov processes. This theme was first taken up by P.Lévy and in particular by S.Bochner when discussing stochastically continuous processes with stationary and independent increments, i.e. *Lévy processes*. Bochner's monograph [32] should be mentioned as the most inspiring source to these considerations. The observation is that every Lévy process $(X_t)_{t \geq 0}$ with state space $\mathbb{R}^n$ is completely determined by one and only one

function $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ which is defined by the relation

$$E(e^{iX_t \cdot \xi}) = e^{-t\psi(\xi)}. \quad (0.1)$$

The function ψ , called the *characteristic exponent* of $(X_t)_{t \geq 0}$, is a continuous negative definite function and contains all information about $(X_t)_{t \geq 0}$. Some results for $(X_t)_{t \geq 0}$ are best proven by looking directly at ψ , for example the process is conservative if and only if $\psi(0) = 0$. But for pathwise considerations it is useful to take the Lévy–Khinchin representation

$$\begin{aligned} \psi(\xi) = & c + ib \cdot \xi + Q(\xi) + \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{-iy \cdot \xi} - \frac{iy \cdot \xi}{1 + |y|^2} \right) \\ & \times \frac{1 + |y|^2}{|y|^2} \mu(dy). \end{aligned} \quad (0.2)$$

Our starting point is the following observation made in [169]. Let $\left((X_t)_{t \geq 0}, P^x\right)_{x \in \mathbb{R}^n}$ be a (nice) Feller process with state space $\mathbb{R}^n$. Then the function

$$q(x, \xi) := - \lim_{t \rightarrow 0} \frac{E^x(e^{i(X_t - x) \cdot \xi}) - 1}{t} \quad (0.3)$$

completely characterizes $\left((X_t)_{t \geq 0}, P^x\right)_{x \in \mathbb{R}^n}$. In analogy to the theory of partial differential operators we will call $q(x, \xi)$ the *symbol of the process* $\left((X_t)_{t \geq 0}, P^x\right)_{x \in \mathbb{R}^n}$. Let us try to understand (0.3) heuristically from two different starting points.

First suppose that $\left((X_t)_{t \geq 0}, P^x\right)_{x \in \mathbb{R}^n}$ is given as a nice Feller process. For fixed $x \in \mathbb{R}^n$ we may consider the random variables X_t under P^x and look at their *characteristic functions*, i.e. we may consider

$$\lambda_t(x, \xi) := E^x(e^{i(X_t - x) \cdot \xi}) = e^{-ix \cdot \xi} E^x(e^{iX_t \cdot \xi}). \quad (0.4)$$

Denoting by $(T_t)_{t \geq 0}$ the semigroup associated with $\left((X_t)_{t \geq 0}, P^x\right)_{x \in \mathbb{R}^n}$ we find

$$T_t u(x) = E^x(u(X_t)) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \lambda_t(x, \xi) \hat{u}(\xi) d\xi, \quad (0.5)$$

which says that $(T_t)_{t \geq 0}$ is a family of pseudo-differential operators and the symbol of T_t is $\lambda_t(x, \xi)$. By assumption $(T_t)_{t \geq 0}$ is a Feller semigroup. Hence

we may look at its generator

$$Au = \lim_{t \rightarrow 0} \frac{T_t u - u}{t}, \quad (0.6)$$

where the limit is taken in the strong sense in the space $C_\infty(\mathbb{R}^n; \mathbb{R})$. Substituting (0.5) into (0.6) we arrive at

$$Au(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} q(x, \xi) \hat{u}(\xi) d\xi \quad (0.7)$$

with $q(x, \xi)$ as in (0.3). Thus the generator of the semigroup (and of the process) is also a pseudo-differential operator. The function $\xi \mapsto \lambda_t(x, \xi)$ must be positive definite (and continuous). Supposing for simplicity for a moment that the process is conservative we have $\lambda_t(x, 0) = 1$ and from the general theory of negative definite functions it follows that $\xi \mapsto -\frac{(\lambda_t(x, \xi) - 1)}{t}$ must be negative definite, which must also hold for the limit. Hence, we find that the symbol $q(x, \xi)$ of the process must be a continuous negative definite function with respect to ξ .

There is another way to understand (0.7) (or (0.3)). As generator of a Feller semigroup A has to satisfy the positive maximum principle, i.e.

$$\sup_{x \in \mathbb{R}^n} u(x) = u(x_0) \geq 0 \text{ implies } Au(x_0) \leq 0. \quad (0.8)$$

In [55] Ph.Courrègue characterizes the operators satisfying the positive maximum principle. In particular he proved under the reasonable assumption that $C_0^\infty(\mathbb{R}^n; \mathbb{R}) \subset D(A)$, then for $u \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ the operator A has the representation (0.7), where $q(x, \xi)$ is a locally bounded function in x which is continuous and negative definite in ξ .

Clearly, the generator completely characterizes the semigroup, and further, the family of characteristic functions $(\lambda_t(x, \xi))_{t \geq 0}$ completely characterize the process. Now we are in the analogous situation as S.Bochner was in case of Lévy processes: The study of the process is reduced to the study of its symbol!

As in the case of Lévy processes it is sometimes more advantageous to use $q(x, \xi)$ directly, but sometimes it is better to use its Lévy-Khinchin decomposition

$$\begin{aligned} -q(x, \xi) = & c(x) + \sum_{j=1}^n b_j(x) \xi_j + \sum_{k,l=1}^n a_{kl}(x) \xi_k \xi_l \\ & + \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{-iy \cdot \xi} - \frac{iy \cdot \xi}{1 + |y|^2} \right) \frac{1 + |y|^2}{|y|^2} N(x, dy). \end{aligned} \quad (0.9)$$

Now we may state the aim of the monograph:

Following Bochner: Construct and study Markov processes by systematically making use of their symbols.

It turns out that in doing so a lot of analysis is to be developed which is not covered by standard theories of pseudo-differential operators, of function spaces etc. This is due to the fact that the symbols under considerations do in general not belong to classical symbol classes. Thus, although stochastic processes are our aim, we have to develop our analytic tools first. We divide our presentation into three parts:

1. Fourier analysis and semigroups.
2. Generators and their potential theory.
3. Markov processes and applications.

For each part we will give a separate introduction.