An Introduction to the
Geometry *of*
Stochastic Flows

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An Introduction to the Geometry *of* Stochastic Flows

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Preface

The aim of the present text is to provide a self-contained introduction to the local geometry of the stochastic flows associated with stochastic differential equations.

The point of view we want to develop is that the local geometry of any stochastic flow is determined very precisely and explicitly by a universal formula referred to as the Chen-Strichartz formula. The natural geometry associated with the Chen-Strichartz formula is the sub-Riemannian geometry whose main tools are introduced throughout the text. By using the connection between stochastic flows and partial differential equations, we apply this point of view to the study of hypoelliptic operators written in Hörmander's form.

Many results contained in this text stem from my stay at the Technical University of Vienna where I had the great pleasure to discuss passionately with Josef Teichmann. I learnt a lot from him and I thank him very warmly. I also would like to take this opportunity to thank Nicolas Victoir for reading early drafts of various parts of the text and for his valuable suggestions.

*F. Baudoin,
Toulouse, June 2004*

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