Experiments in Undergraduate Mathematics

A Mathematica-Based Approach

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EXPERIMENTS IN UNDERGRADUATE MATHEMATICS

A Mathematica-Based Approach

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<http://www.icbl.hw.ac.uk/tltp/>.

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David Klug, Richard Templer and Ian Gould, lecturers in the Chemistry Department at Imperial College, made the bold decision to introduce *Mathematica* into their first-year course on mathematics. They provided us with our most important testing ground for evaluating prototype versions of the mathematical experiments in this book. We thank them, and the many undergraduate students, in the Chemistry department and elsewhere, who worked on the experiments and provided us with invaluable information and insights.

Finally, we wish to thank Dan Moore for all the detailed discussions about Chapter 11.

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Preface

This book takes its place amongst a growing number about using a computer mathematics system* (CMS) as an environment for learning mathematics. These systems are already changing both the way mathematics is done and the way mathematics is used. We can be sure, then, that they will have a role in the way mathematics is learnt and taught. But they have been around for a relatively short time, and the pace of curricular change is slow; in any case, there is still a lively debate among educators about what the scale and nature of their role will be, or ought to be.

We believe that a CMS allows us to present mathematics to students as an *experimental* subject. The term “experiment” that we use in this book is not used casually or rhetorically. In a laboratory experiment, in clear contrast to lectures or tutorials, the student, though perhaps guided by a lab assignment script, can exert a measure of *control* over the process. Experiments, too, are often collaborative, in the sense that students can pool their ideas and energies; such interactions are less common in traditional mathematics learning. Finally, and most importantly, an approach based in part upon experiment encourages the student to formulate *mathematical conjectures*. This conjecturing process is central to mathematics and to mathematical problem-solving, but has traditionally played only a minor role in the mathematical activity of students.

Our approach is related to the doctrine of *constructivism*: the idea that conceptual learning involves the learner building, bit-by-bit, his or her own understanding of mathematical knowledge. A consequence of this idea is that, in mathematics at least, mere *instruction* is unlikely to be effective: at any rate, not on its own. Constructivism emphasises the importance of cognitively challenging mathematical *activity*.

* We prefer this term to the more traditional “Computer Algebra System”, which fails to convey the power and scope of modern mathematical software.

The topics covered in this book roughly correspond to those traditionally studied in the A-level school syllabus in England and Wales, though recent and continuing curriculum changes mean that several topics are now studied for the first time at university. The first year undergraduate with mathematics 'A' level will find this book most useful for revision: for strengthening his or her understanding of topics first met at school. In North American terms, this book is suitable for freshman calculus.

We welcome your comments and thoughts on the contents of this book. Send them along to:

The METRIC Project
Mathematics Department, Imperial College
London SW7 2BZ
England.

Electronic mail: **`metric-proj@ic.ac.uk`**

World-Wide-Web: **`http://metric.ma.ic.ac.uk/Message.html`**

Phillip Kent, Phil Ramsden and John Wood — May 1996.

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To the learner

This is a book about *doing* mathematics. Mathematics is not a subject you can learn just by reading about it — though that can sometimes make you *think* you've learned it. Learning mathematics means getting your hands dirty: getting to grips with the subject directly. What's different about this book is that you're invited to do mathematics armed with *Mathematica*. When you see what *Mathematica* can do for you, you'll understand what an important difference that is.

Most of the “doing” in this book happens in the *mathematical experiments*. Though these sometimes take up less space on the page than what we've called the Reading, they are not less important. On the contrary: they're what this book is really all about.

The structure of this book

The book is divided into five chapters. The first (*Getting Started*) is a quick introduction to the *Mathematica* software. Then comes *Foundations*, basic topics on which the following ones rest. The three main chapters, *Calculus*, *Vectors & Matrices* and *Complex Numbers*, can then be studied independently of one another.

Each **topic** is broken down into a number of **modules** (one, two or three), each containing material for approximately one hour of study at the computer (*Differentiation*, for example, has two modules). The title page of each chapter lists the contents of all its modules.

Each module contains about a half dozen **experiments**, indicated by a shaded background, which consist of a set of activities to be carried out at the computer. Almost every experiment has a section of **Preparatory Reading**, to be read before you start, and some have **Post-experiment Reading**, to be read when you have finished.

Practice questions

Placed at appropriate points in each module you'll find **practice questions**. These are an electronic version of those you find in ordinary mathematics textbooks.

There is a difference, though. In a textbook, the questions are fixed. But the practice questions, and their answers, are generated by the computer. Each time you set yourself a question, it will be slightly different from the last time. The system makes no judgements and it does not keep scores.

Mathematics and *Mathematica*

This book is for learning or revising mathematics, not for becoming a *Mathematica* expert. Of course, you will end up learning quite a bit about *Mathematica*, and that skill will always be useful to you. And doing *Mathematica* counts as doing mathematics, because *Mathematica* is that kind of program. Actually, you may well become unsure where *Mathematica* ends and mathematics begins; most *Mathematica* users get like that eventually.

By the way: if you find yourself getting interested in more advanced uses of the program, we have devised some *Additional Activities*. These can be found on our World-Wide-Web site (<http://metric.ma.ic.ac.uk/>).

Finally, we'll quote a passage from the "advice to the learner" section of another book* which we like very much and which we think goes well here:

"If I am asked to solve a problem, or do something on the computer, and I am not told how to do it, what should I do?" The answer is very simple. Make some guesses (better yet, conjectures) and try them out on the computer. Ask

* *Learning Abstract Algebra with ISETL*, Ed Dubinsky and Uri Leron, Springer-Verlag, 1994. Page xii.

yourself if it worked — or what part of your guess worked and what part did not. Try to explain why. Then refine your guess and try again. And again. Keep repeating this cycle until you understand what is going on. The most important thing for you to remember is not to think of these explorations in terms of success and failure. Whenever the computer result is different from what you expected, think of this as an opportunity for you to improve your understanding. Remember: instead of just being stuck, not knowing what to do next, you now have an opportunity to experiment, to make conjectures and try them out, and to gradually refine your conjectures until you are satisfied with your understanding of the topic at hand.

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To the lecturer

We envisage three ways in which this book might be used:

- as an integral part of a course based around computer laboratory sessions, with or without lectures.
- as a supplement to either a conventional lecture course, or a lab course using another set of *Mathematica*-based materials.
- as revision material for mathematics required for a course that you are teaching.

The book does not assume any previous experience with *Mathematica* or similar software on the part of learners. Neither does it set out to teach *Mathematica* explicitly. However, the authors' experience in piloting the contents of this book suggests that learners can, and do, become very skilled *Mathematica* users quite quickly.

The METRIC software

We have provided special *Mathematica* functions for tasks where working from scratch would be too cumbersome, or too intricate for a novice user. These functions work with the standard *Mathematica* syntax; we have not attempted to implement a simplified, novice version. The standard syntax, though not transparent, is inherently well designed and naturally mathematical in structure, and supports students' mathematical thinking.

We have also provided computer-generated, randomised *practice questions*. Students can set themselves such questions at any time, but are invited to do so at specific points in the text.

Our software contains a *logging mechanism* that allows students' sessions with *Mathematica*, or selected parts of those sessions, to be automatically recorded. Full directions for activating this mechanism are to be found in the text file **LOGGING.TXT** in the METRIC package.

Computer labs and orientation sessions

In our experience, even students completely new to *Mathematica* can become quite proficient users provided they are given two or three hours of “orientation” in a computer lab with teacher/instructor support. The module *Getting Started* is designed particularly for such sessions, with *Functions & Graphs* a natural second module to follow.

Adapting the materials

You will, of course, be able to adapt the curricular materials in this book for your own purposes, for example by selecting particular experiments, or parts of experiments. We hope, too, that you will feel free to modify the special functions for your own uses, and perhaps improve them.

You are also invited to make free use of the Practice Question code, and to author your own questions. This is easier than it sounds; the design of the practice questions has been deliberately kept simple for this very purpose. Instructions on how to write practice questions are to be found in the text file **PRACTICE.TXT** in the METRIC package.

We would be pleased to hear about any modifications that you do carry out. Please contact us at the addresses given in the Preface.

Additional resources

Please check our World-Wide-Web site (<http://metric.ma.ic.ac.uk/>) for additional materials that we will be making available from time to time.

The software for this book

To perform the experiments in this book you will need *Mathematica* version 2.0 or later with the notebook front end, together with the METRIC software package.

The METRIC package consists of:

- *Mathematica* custom functions that are used throughout the experiments;
- practice questions, and the *Mathematica* code for running them;
- code for logging students' *Mathematica* sessions;
- text files containing detailed technical instructions, including directions for installation.

Disk access

The hardcover edition of this book includes a floppy disk (DOS format) with the complete set of programs.

You can also get the floppy disk by mail by writing to:

The METRIC Project (EUM Disk)
Mathematics Department
Imperial College
London SW7 2BZ
England.

The cost is £15 sterling. Please send a cheque payable to "Imperial College" and drawn on a British bank.

Internet access

The METRIC package is freely available on the World-Wide Web, at the URL
<http://metric.ma.ic.ac.uk/>

You may also use anonymous FTP or Gopher to **metric.ma.ic.ac.uk** (IP address **155.198.192.26**). The programs are available at the *MathSource* archive in the USA:

<http://www.wolfram.com/mathsource/>

Platforms and formats

The floppy disk contains three archive files for DOS, Macintosh and Unix:

mathetic.exe, **mathetic.sea** and **mathetic.tar**, respectively. There is also a directory, **source**, containing the files in normal form. Installation instructions are given in the file **INSTALL.TXT**.

Updates and on-line information

For the latest updates to the software and learning modules, please visit our WWW site at:

<http://metric.ma.ic.ac.uk/>

or use anonymous FTP or Gopher to **metric.ma.ic.ac.uk** (IP address **155.198.192.26**).

Mathematica version 3.0

As we go to press, *Mathematica* version 3.0 has not yet been released. It seems likely that the directory structure will change somewhat, and therefore that installation instructions will have to be modified slightly. We shall post information on this at the above WWW site.