Population Biology and Criticality

From Critical Birth-Death Processes
to Self-Organized Criticality in
Mutation Pathogen Systems

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Preface

This book describes novel theories of mutation pathogen systems showing critical fluctuations as a paradigmatic example of an application of the mathematics of critical phenomena to the life sciences. It will enable the reader to understand the implications and future impact of these findings, and at same time allow them to actively follow the mathematical tools and scientific origins of critical phenomena. Hopefully this will lead to further fruitful applications of the mathematics of critical phenomena in other fields of the life sciences.

We show through explicit calculations the basis for understanding critical fluctuations as a genuine effect of stochastic dynamic systems, including the original background in statistical physics. Birth and death processes occur in many applications in the life sciences, in ecological and epidemiological systems and cancer growth, for example. In such simple systems the critical threshold appears and can be analyzed in all its different aspects.

Guided by our own research experience we then proceed to a system of pathogens evolving towards a critical state in a multi-mutant system. Although we concentrate on a special case study, the mechanism will certainly have much wider implications theoretically, understanding the mechanisms leading to systems with large fluctuations, as well as empirically, as in many real world systems such large fluctuations are observed but their mechanism often not recognized as such or not well understood.

To enable researchers in the various fields of application to analyze their systems appropriately we show explicitly the calculations and mathematical tools involved, which are now widespread in the textbook literature and the original scientific literature in journals. Since they are often spread over a wide range of scientific fields they use very different language and notations.

From our own teaching experience we see the necessity of taking the reader from basic notions of stochastic dynamics, often taught to physics students and probability theoreticians but rarely to students from the life sciences, to a detailed analysis of the birth and death process, which is often taught to biologists and other life scientists, but rarely in the aspect of criticality. The field of critical phenomena in birth and death processes itself is a branch of rather high specialization in statistical physics with loose applications mainly in condensed matter physics or no application directly in mind.

However, when looking at real world data, large fluctuations are often observed, and cannot always be traced back to large external influences, hence the need to search for possible internal mechanisms. These internal mechanisms are given by the theory of critical phenomena, as we will explicitly demonstrate in our paradigmatic case study.

This book contains two parts. The first includes an introduction to stochastic dynamic systems and criticality, the second explores the application of mutation pathogen systems as a case study for wider applications.

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N. Stollenwerk and V. Jansen
Lisbon and London, August 2009

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