THE PRINCIPLES OF NEWTONIAN AND QUANTUM MECHANICS

The Need for Planck's Constant, h

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The Need for Planck's Constant, h

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To Charlyne,
with all my love

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FOREWORD BY BASIL HILEY

One of the perennial problems in the continued specialization of academic disciplines is that an important but unexpected result in one area can go completely unnoticed in another. This gap is particularly great between theoretical physics and the more rigorous mathematical approaches to the basic formalism employed by physicists. The physicists show little patience with what to them seems to be an obsession with the minute detail of a mathematical structure that appears to have no immediate physical consequences. To mathematicians there is puzzle that sometimes borders on dismay at some of the 'vague' structures that physicists use successfully. In consequence, each group can be totally unaware of the important progress made by the other. This is not helped by the development of specialised technical languages, which can prevent the 'outsider' seeing immediately the relevance of these advances. At times, it becomes essential to set down these advances in a way that brings the two groups together. This book fits into this category as it sets out to explain how recent advances in quantization procedures for Lagrangian manifolds has relevance to the physicist's approach to quantum theory.

Maurice de Gosson has considerable mathematical expertise in the field of Lagrangian quantization, which involves a detail study of symplectic structures, the metaplectic covering of these structures and Maslov indices, all topics that do not fall within the usual remit of a quantum physicist. It is a mathematicians attempt to show the precise relationship between classical and quantum mechanics. This relationship has troubled physicists for a long time, but in spite of this, the techniques presented in this book are not very familiar to them. They are generally content with the plausible, but somewhat vague notion of the correspondence principle. However any detailed analysis of the precise meaning of this principle has always been beset with problems. Recently decoherence has become a fashionable explanation for the emergence of

the classical world even though it, too, has its difficulties. This book provides an alternative and more mathematically rigorous approach of the relationship between the classical and quantum formalisms.

Unsurprisingly the discussion of classical mechanics takes us into a detailed study of the symplectic group. A notable feature of this discussion is centred on Gromov's 'non-squeezing' theorem, which although classical, contains the seeds of the uncertainty principle. The common conception of Liouville's theorem is that under a symplectic transformation a volume in phase space can be made as thin as one likes provided the volume remains constant. Thus, it would be possible to pass the proverbial camel through the eye of a needle no matter how small the eye! This is in fact not true for the 'symplectic camel'. For a given process in phase space, it is not, repeat not, possible to shrink a cross-section defined by conjugate co-ordinates like x and p_x to zero. In other words, we have a minimum cross-sectional area within a given volume that cannot be shrunk further. It is as if the uncertainty principle has left a 'footprint' in classical mechanics.

Perhaps the most important topic discussed in the book is the role of the metaplectic group and the Maslov index. Apart from the use of this group in optics to account for phenomenon like the Gouy phase, the metaplectic group is almost a complete stranger to the physics community, yet it is the key to the relationship between classical and quantum mechanics. Indeed, it is argued here that Schrödinger's original derivation of his famous equation could be regarded as the discovery of the metaplectic representation of the symplectic group.

To understand how this comes about we must be aware of two facts. First we must realise that the metaplectic group double covers the symplectic group. This is exactly analogous to the double cover of the orthogonal group by the spin group. In this sense, it can be regarded as the 'spin group' for the symplectic group. Secondly, we must discuss classical mechanics in terms of the Hamiltonian flow, f_t , which is simply the family of symplectic matrices generated by the Hamiltonian. In contrast, the time evolution in quantum mechanics is described by the Hamiltonian through the group of unitary operators U_t . What this book shows is that the lift of f_t onto the covering space is just U_t ! This is a remarkable result which gives a new way to explore the relationship between classical and quantum mechanics.

Historically it was believed that this procedure only applied to Hamiltonians that were at most quadratic in position and momentum. This limitation is seen through the classic Groenewold-van Hove 'no-go' theorem. However,

this lift can be generalised to all Hamiltonians by using an iteration process on small time lifts. This approach has similarities with the Feynman path integral method and it is based the Lie–Trotter formula for flows. It has the advantage over the Feynman approach in that it is not a “sum over (hypothetical) paths”, but is a mathematically rigorous consequence of the metaplectic representation, together with the rule This opens up the possibilities of new mathematical questions concerning the existence of generalised metaplectic representations, a topic that has yet to be addressed in detail.

All of this opens up a new mathematical route into quantum theory offering a much clearer relation between the classical and the quantum formalisms. As the approach is mathematical, there is no need to get embroiled in the interminable debate about interpretations of the formalism. Indeed, because of this focus on the mathematics without any philosophical baggage, it is possible to see exactly how the Bohm approach fits into this general framework, showing the legitimacy of this approach from a mathematical point of view. Indeed, we are offered further insights into this particular approach, which I find particularly exciting for obvious reasons. I hope others will be stimulated into further explorations of the general structure that Maurice de Gosson unfolds in this volume. I am sure this structure will reveal further profound insights into this fascinating subject.

Basil Hiley, *Birkbeck College, London, 2001.*

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PREFACE

The aim of this book is to expose the mathematical machinery underlying Newtonian mechanics and two of its refinements, semi-classical and non-relativistic quantum mechanics. A recurring theme is that these three Sciences are all obtained from a single mathematical object, the Hamiltonian flow, viewed as an abstract group. To study that group, we need symplectic geometry and analysis, with an emphasis on two fundamental topics:

Symplectic rigidity (popularly known as the “principle of the symplectic camel”). This principle, whose discovery goes back to the work of M. Gromov in the middle of the 1980’s, says that no matter how much we try to deform a phase-space ball with radius r by Hamiltonian flows, the area of the projection of that ball on a position-momentum plane will never become inferior to πr^2 . This is a surprising result, which shows that there is, contrarily to every belief, a “classical uncertainty principle”. While that principle does not contradict Liouville’s theorem on the conservation of phase space volume, it indicates that the behavior of Hamiltonian flows is much less “chaotic” than was believed. Mathematically, the principle of the symplectic camel shows that there is a symplectic invariant (called Gromov’s width or symplectic capacity), which is much “finer” than ordinary volume. Symplectic rigidity will allow us to define a semi-classical quantization scheme by a purely topological argument, and will allow us to give a very simple definition of the Maslov index without invoking the WKB method.

The *metaplectic representation* of the symplectic group. That representation allows one to associate in a canonical way to every symplectic matrix exactly two unitary operators (only differing by their signs) acting on the square integrable functions on configuration space. The group $Mp(n)$ of all these operators is called the metaplectic group, and enjoys very special properties; the most important from the point of view of physics since it allows

the explicit resolution of all Schrödinger's equations associated to quadratic Hamiltonians. We will in fact partially extend this metaplectic representation in order to include even non-quadratic Hamiltonians, leading to a precise and mathematically justifiable form of Feynman's path integral.

An important issue that is addressed in this book is that of quantum mechanics in phase space. While it is true that the primary perception we, human beings, have of our world privileges positions, and their evolution with time, this does not mean that we have to use only, mathematics in configuration space. As Basil Hiley puts it "*...since thoughts are not located in space-time, mathematics is not necessarily about material things in space-time*". Hiley is right: it is precisely the liberating power — I am tempted to say the *grace* — of mathematics that allows us to break the chains that tie us to one particular view of our environment. It is unavoidable that some physicists will feel uncomfortable with the fact that I am highlighting one unconventional approach to quantum mechanics, namely the approach initiated by David Bohm in 1952, and later further developed by Basil Hiley and Bohm himself. To them I want to say that since this is *not* a book *on* the epistemology or ontology of quantum mechanics (or, of physics, in general), I had no *états d'âme* when I used the Bohmian approach: it is just that this way of seeing quantum mechanics is the easiest way to relate classical and quantum mechanics. It allows us to speak about "particles" even in the quantum regime which is definitely an economy of language... and of thought! The Bohmian approach has moreover immediately been well-accepted in mathematical circles: *magna est veritas et praevalebit...*

While writing this book, I constantly had in mind two categories of readers: my colleagues — mathematicians, and my dear friends — physicists. The first will, hopefully, learn some physics here (but presumably, not the way it is taught in usual physics books). The physicists will get some insight in the beautiful unity of the mathematical structure, symplectic geometry, which is the most natural for expressing both classical and quantum mechanics. They will also get a taste of some sophisticated new mathematics (the symplectic camel, discussed above, and the Leray index, which is the "mother" of all Maslov indices). This book is therefore, in a sense, a tentative to reconcile what Poincaré called, in his book *Science and Hypothesis*, the "two neighboring powers": Mathematics and Physics. While Mathematics and Physics formed during centuries a single branch of the "tree of knowledge" (both were parts of "natural philosophy"), physicists and mathematicians started going different ways during the last century (one of the most recent culprits being the Bourbaki school). For instance, David Hilbert is reported to have said that

“Physics is too difficult to leave to physicists”, while Albert Einstein characterized Hilbert’s physics (in a letter to Hermann Weyl) as “infantile”. To be fair, we must add that Einstein’s theory was really based on physical principles, while Hilbert’s travail in physics was an exercise in pure mathematics (we all know that even today many mathematical texts, which claim to be of physical interest, are too often just pure mathematics dressed up in a phony physical language).

A few words about the technical knowledge required for an optimal understanding of the text. The mathematical tools that are needed are introduced in due time, and are rather elementary (undergraduate linear algebra and calculus suffice, together with some knowledge of the rudiments of the theory of differential forms). This makes the book easily accessible to a rather large and diversified scientific audience, especially since I tried as much as possible to write a “self-contained” text (a few technical Appendices have been added for the reader’s convenience). A word to my colleagues – mathematicians: this book can be read without any particular prior knowledge of physics, but it is perhaps somewhat unrealistic to claim that it is an introduction “from scratch” to the subject. Since I have tried to be intelligible by both mathematicians and physicists, I have made every effort to use rigorous, but simple mathematics. I have, however, made every effort to avoid Bourbachian *rigor mortis*.

This book is structured as follows:

Chapter 1 is devoted to a review of the basic principles of Newtonian and quantum mechanics, with a particular emphasis on its Bohmian formulation, and the “quantum motion” of particles, which is in a sense simpler than the classical motion (there are no “caustics” in quantum mechanics: the latter only appear at the semi-classical level, when one imposes classical motion to the wave functions).

Chapter 2 presents modern Newtonian mechanics from the symplectic point of view, with a particular emphasis on the Poincaré-Cartan form. The latter arises in a natural way if one makes a certain physical hypothesis, which we call, following Souriau, the “Maxwell principle”, on the form of the fundamental force fields governing the evolution of classical particles. The Maxwell principle allows showing, using the properties of the Poincaré-Cartan invariant, that Newton’s second law is equivalent to Hamilton’s equations of motion for these force fields.

In *Chapter 3*, we study thoroughly the symplectic group. The symplectic group being the backbone of the mathematical structure underlying Newtonian mechanics in its Hamiltonian formulation, it deserves as such a thorough study

in its own right. We then propose a semi-classical quantization scheme based on the principle of symplectic rigidity. That scheme leads in a very natural way to the Keller-Maslov condition for quantization of Lagrangian manifolds, and is the easiest way to motivate the introduction of the Maslov index in semi-classical mechanics.

In *Chapter 4*, we study the so fundamental notion of action, which is most easily apprehended by using the Poincaré-Cartan invariant introduced in Chapter 2. An important related notion is that of generating function (also called Hamilton's "two point characteristic functions"). We then introduce the notion of Lagrangian manifold, and show how it leads to an intrinsic definition of the phase of classical completely integrable systems, and of all quantum systems.

Chapter 5 is devoted to a geometrical theory of semi-classical mechanics in phase space, and will probably be of interest to theoretical physicists, quantum chemists and mathematicians. This Chapter is mathematically the most advanced, and can be skipped in a first reading. We begin by showing how the Bohmian approach to quantum mechanics allows one to interpret the wave function as a half-density in phase space. In the general case, wave forms are (up to a phase factor) the square roots of de Rham forms defined on the graph of a Lagrangian manifold. The general definition of a wave form requires the properties of Leray's cohomological index (introduced by Jean Leray in 1978); it is a generalization of the Maslov index, which it contains as a "byproduct". We finally define the "shadows" of our wave forms on configuration space: these shadows are just the usual semi-classical wave functions familiar from Maslov theory.

Chapter 6 is devoted to a rather comprehensive study of the metaplectic group $Mp(n)$. We show that to every element of Mp (we can associate an integer modulo 4, its Maslov index, which is closely related to the Leray index. This allows us to eliminate in a simple and elegant way the phase ambiguities, which have been plaguing the theory of the metaplectic group from the beginning. We then define, and give a self-contained treatment, of the inhomogeneous metaplectic group $IMp(n)$, which extends the metaplectic representation to affine symplectic transformations. We also discuss, in a rather sketchy form, the difficult question of the extension of the metaplectic group to arbitrary (non-linear) symplectic transformations, and Groenewold-Van Hove's famous theorem.

The central theme of *Chapter 7* is that although quantum mechanic cannot be derived from Newtonian mechanics, it nevertheless emerges from it via the

theory of the metaplectic group, provided that one makes a physical assumption justifying the need for Planck's constant h . This "metaplectic quantization" procedure is not new; it has been known for decades in mathematical circles for quadratic Hamiltonians. In the general case, there is, however an obstruction for carrying out this quantization, because of Groenewold-Van Hove's theorem. This theorem does however not mean that we cannot extend the metaplectic group to non-quadratic Hamiltonians. This is done by using the Lie-Trotter formula for classical flows, and leads to a general metaplectic representation, from which Feynman's path integral "pops out" in a much more precise form than in the usual treatments.

The titles of a few Sections and Subsections are followed by a star * which indicates that the involved mathematics is of a perhaps more sophisticated nature than in the rest of the book. These (sub)sections can be skipped in a first reading.

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Maurice de Gosson, *Blekinge Institute of Technology, Karlskrona*,
March 2001

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