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ALGORITHMICS OF MATCHING UNDER PREFERENCES

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With a foreword by Kurt Mehlhorn

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To Carolyn, Susan and Paul

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Preface

Matching problems involving preferences occur in widespread applications such as the assignment of children to schools, school-leavers to universities, junior doctors to hospitals, students to campus housing, kidney transplant patients to donors and so on. The common thread is that agents have preferences over the possible outcomes and the task is to find a *matching* (i.e., an assignment of the participants to one another) that is in some sense optimal with respect to these preferences.

These problems are growing in importance in an era in which more and more elements of society are embracing diverse forms of electronic communication, and individuals are increasingly used to making choices via the internet. The ease by which preference information can now be collected has contributed to the growing tendency for matching processes to be centralised. Due to the typical size of applications (for example, in China, over 10 million students apply for admission to higher education annually through a centralised process), trying to construct optimal allocations manually (given a suitable definition of “optimal”) is simply not feasible.

Thus algorithms are required to automate the process of constructing optimal matchings. Again, due to the size of typical applications, the efficiency of the algorithms is of paramount importance. The notion of optimality is also a key consideration: many matching processes are conducted by publicly-funded organisations, and there is an increasing tendency for the decisions reached by these organisations to be scrutinised both in the media and by individuals through Freedom of Information requests, for example. Thus the algorithms need to construct matchings that are not just provably optimal, but also are seen to be “fair” by the agents involved.

This book focuses on algorithmic aspects of matching problems involving preferences — our aim is to describe efficient (polynomial-time

algorithms) that produce optimal matchings (under many different notions of optimality) or to highlight complexity results that imply the non-existence of such algorithms. We also describe some of the many applications in which these algorithms are used.

Our interest, therefore, is in the underlying computational matching problems that arise in matching markets. The importance of this research area was recognised by the award, in 2012, of the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel (commonly known as the Nobel Prize in Economic Sciences) to Alvin Roth and Lloyd Shapley, who are both leading figures in the research area.

The archetypal matching problem involving preferences is the celebrated Stable Marriage problem, first introduced by David Gale and Lloyd Shapley in 1962 [235]. The main contribution of this paper was an algorithm, known as the Gale–Shapley algorithm, to solve this problem. This algorithm has been put to practical use in a wide-range of large-scale applications in countries throughout the world.

Thereafter, Donald Knuth’s interest in the problem was reflected in a collection of lectures published in 1976 [394]. A series of papers written by Dan Gusfield, Rob Irving and co-authors in the 1980s uncovered deep structural relationships that are present in Stable Marriage problem instances, and showed how these lead to efficient algorithms for many problems associated with the computation of so-called stable matchings. This culminated in the publication of their monograph “The Stable Marriage Problem: Structure and Algorithms” in 1989 [261].

This book has become the standard reference in the literature for structural and algorithmic aspects of the Stable Marriage problem, and indeed its non-bipartite generalisation, the Stable Roommates problem. It contains clear and elegant descriptions of a range of stable matching algorithms, showing how these exploit the fundamental correspondence between the lattice structure of stable matchings and the so-called rotation poset that holds in problem instances. It is a much valued resource for those engaged in stable matching research.

Shortly afterwards, in 1990, a related monograph, entitled “Two-sided matching: a case study in game-theoretic modeling and analysis”, by Al Roth and Marilda Sotomayor, was published. This book focused mainly on game-theoretic and strategic questions involving the Stable Marriage problem and its many-one generalisation, the Hospitals / Residents problem, among others.

Since the publication of these books, the algorithmic interest in the Stable Marriage problem and its variants has shown no sign of diminishing in the literature. Papers since then have also focused on the Stable Roommates problem, the Hospitals / Residents problem and the House Allocation problem.

We classify matching problems with preferences according to three broad classes: (i) bipartite matching problems with two-sided preferences (including the Stable Marriage and Hospitals / Residents problems and their variants), (ii) bipartite matching problems with one-sided preferences (incorporating the House Allocation problem and its variants), and (iii) non-bipartite matching problems with preferences (including the Stable Roommates problem and its variants). Problems in classes (i) and (iii) will mainly be studied with respect to the stability of a given matching, whilst problems in class (ii) involve other optimality criteria such as Pareto optimality, popularity and rank-maximality.

Researchers who are interested in matching problems with preferences are drawn from a range of backgrounds, including the algorithms and complexity, discrete mathematics, game theory and economics, and computational social choice fields. The community is growing and its strength is reflected in the fact that several workshops in the area have taken place in recent years.

The year 2008 saw the first workshop on matching problems with preferences, with an emphasis on algorithmic aspects, co-organised by the author, Magnús Halldórsson, Rob Irving and Kazuo Iwama. This one-day meeting, entitled “MATCH-UP: Matching Under Preferences – Algorithms and Complexity” [266], took place in Reykjavík as a satellite workshop of ICALP 2008 (the 35th International Colloquium on Automata, Languages and Programming). Invited speakers included Kurt Mehlhorn, Al Roth and Marilda Sotomayor. David Gale, who had agreed to open the workshop with an invited lecture, sadly died in the March preceding the workshop; consequently the event was dedicated to his memory and Marilda Sotomayor’s invited lecture formed a tribute to his career and their collaboration. Selected papers from the workshop, and several other papers on matching under preferences, appeared in a special issue of the journal *Algorithmica* [417], published in 2010.

A conference entitled “Frontiers of Market Design” was organised by Bettina Klaus in Ascona, Switzerland, in May 2012. Its main focus was on matching markets, and this included matching problems with preferences. Furthermore, a mini-symposium on “Matchings with Preferences”

was organised by Christine Cheng at the SIAM Conference on Discrete Mathematics, in Halifax, Canada, in June 2012.

A successor to the 2008 MATCH-UP workshop, entitled “MATCH-UP 2012: the Second International Workshop on Matching Under Preferences” took place, this time over two days, in Budapest in July 2012. This workshop was co-organised by Péter Biró, the author, Tamás Fleiner and Tamás Solymosi, and aimed to involve economists as well as computer scientists. On this occasion, the invited speakers were Nicole Immorlica, Rob Irving, Fuhito Kojima, and Tayfun Sönmez. The workshop was very well-attended, with 68 participants. Again, several papers appearing in the proceedings [78] were invited for submission to a special issue of the journal *Algorithms*.

October 2012 saw the announcement of the award of the Nobel Prize in Economic Sciences to Al Roth and Lloyd Shapley, as previously mentioned, for their work in “the theory of stable allocations and the practice of market design”. This reflects both Shapley’s contribution to the Stable Marriage algorithm among other theoretical advances, and Roth’s application of these results to matching markets involving the assignment of junior doctors to hospitals, pupils to schools and kidney patients to donors (through “kidney exchanges”). The Stable Marriage problem and its variants essentially form the basis of Part 1 of this book, whilst the many applications of matching problems will be highlighted throughout.

These events serve to illustrate the level of interest in the research community in matching under preferences. Indeed at the 2008 MATCH-UP workshop, it became clear from conversations with participants that the time was right for a comprehensive update of the state of the art concerning algorithmic aspects of matching under preferences. This book aims to provide such an update. Whilst in a sense it could be regarded as a “sequel” to Gusfield and Irving’s monograph [261], our aim is to expand the range of matching problems that they considered and additionally include alternative optimality criteria besides stability.

The intended readership of this book includes PhD students, postdoctoral researchers and academic staff engaged in research on matching under preferences, senior undergraduate and taught postgraduate students engaged in project work relating to matching under preferences or taking an advanced course on matching theory, and indeed administrators of centralised matching schemes who are interested in the algorithms that underpin these programmes.

The book is largely a comprehensive, classified and guided survey through the literature on matching problems with preferences. Thus the

majority of results have been published already in some form. However the book does include some new results due to the author (or his collaborators, reproduced with their permission) that have not been published previously.

Although much effort has been taken to check the correctness of the assertions made in what follows, in a project of this magnitude it is inevitable that there will be errors. For these I offer my apologies now. The careful reader who does discover an error is invited to send the details to david@optimalmatching.com. A list of corrections will be made available via <http://www.optimalmatching.com/AMUP>.

DFM

Glasgow, October 2012

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Foreword

Matching under preferences is a topic of great practical importance, deep mathematical structure, and elegant algorithmics. The great practical importance stems from the numerous applications such as the assignment of students to universities, families to housing, kidney transplant patients to donors, and so on. In these applications, the participants have preferences over the outcomes and the goal is to find an assignment that optimises the satisfaction of the participants. The importance of the area was recently clearly demonstrated to the world-at-large by the award of the 2012 Nobel Prize in Economics to Alvin Roth and Lloyd Shapley, partly for their work on the stable marriage problem.

Instances may be very large. For example, in China, over 10 million students apply for admission to higher education annually through a centralised process. Thus efficient algorithms are needed. In the past two decades, matching under preferences has become a hot topic in algorithms research with many new structural results, improved algorithms, and new workshops dedicated to the area.

I can attest to the beauty of the subject from my own experience. After hearing a talk by David Manlove on “Strong stability in the Hospitals / Residents Problem” at the British Colloquium for Theoretical Computer Science in 2003, I became interested in matching problems involving preferences. Indeed, some of my results feature in what follows.

This book covers the research area in its full breadth and beauty. It includes most of the recent results in a coherent presentation. Written by one of the foremost experts in the area, it is a timely update to “The Stable Marriage Problem: Structure and Algorithms” (D. Gusfield and R.W. Irving, 1989).

This book will be required reading for anybody working on the subject; it has a good chance of becoming a classic.

Kurt Mehlhorn

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Acknowledgments

There are a lot of people to whom I owe a debt of gratitude in connection with the writing of this book. In some cases, this book simply would not have come about were it not for them.

Firstly, I would like to thank Rob Irving for introducing me to the area of stable matching problems, and subsequently to other matching problems involving preferences. I learned of his expertise in this area as a PhD student under his supervision, but it was not until 1998, when I worked on the “Stable Matching Algorithms” research project (funded by the Engineering and Physical Sciences Research Council) that we started to work together on these problems.

Collaborating with Rob allowed me to quickly discover that studying matching problems with preferences from an algorithmic perspective is a very attractive area to work in for several reasons: (i) many of the problems have direct practical applications; (ii) the problems themselves are engaging and in some cases quite straightforward to specify (and thus accessible to non-specialists), yet challenging to solve; (iii) there are many rich and elegant mathematical structures that can be shown to hold in problem instances, which in turn can be exploited by efficient algorithms; and (iv) although much work has been done on these problems, there are still many interesting open problems to pursue. I gained much from Rob’s knowledge, insight and enthusiasm in connection with this research area, which in turn gave me the opportunity to carve out my own research agenda. I have been very fortunate to have had Rob as a colleague and friend, and I hope that our collaboration will continue for many years.

The idea of writing this book was due to Kazuo Iwama. I am very grateful to him for inviting me in early 2008 to make this monograph a contribution to World Scientific’s series on Theoretical Computer Science.

Work on the book began in earnest in summer 2008, and therefore it has been a long-running project. I would like to thank Iwama-*san* for his patience over the years whilst I juggled the writing of this book with other research activities, in addition to teaching and administrative duties.

Similarly I would like to thank all those at World Scientific who have been involved in the production of this book for their assistance and patience, including the editors Quan Liang, Yeow-Hwa Quek and Chin Ang Ng. I would also like to thank Rajesh Babu for technical assistance with L^AT_EX. I have been especially grateful for the publisher's flexibility with deadlines. One of the challenges with this project running over several years is that chapters written some time ago have had to have be constantly updated to take account of the most recent developments.

Within the School of Computing Science at the University of Glasgow, three colleagues in particular have helped me to bring this project to a successful conclusion. Firstly Joe Sventek, the Head of School, kindly granted me a sabbatical from 1 January 2012 to 30 June 2012, which was especially productive. This was due in no small part to Helen Purchase, who took over my duties as Class Head for our Level 4 undergraduate students with great efficiency during this period. Also Rob Irving came out of retirement to cover the Algorithmics course that I usually teach in the second semester.

In terms of the technical writing, there are again many people who have assisted in some way, and who deserve recognition. Rob Irving, once again, has assisted with the task of proof-reading each chapter, which has been very valuable. Also Péter Biró has been a great source of advice in connection with the decentralised algorithms covered in Sec. 2.6 and Sec. 4.3.3. However I stress that any remaining errors are entirely my responsibility.

I would also like to express my gratitude to Kurt Mehlhorn for kindly agreeing to write the Foreword of this book, and for his support of this project in its early stages.

Others who have helped in ways such as providing sources from the literature, answering my queries or commenting on draft sections, or providing new observations (which are acknowledged again at the relevant points) include Haris Aziz, Christine Cheng, Pavlos Eirinakis, Tamás Fleiner, Hal Gabow, Dan Hirschberg, Chien-Chung Huang, Elena Inarra, Augustine Kwanashie, Iain McBride, Eric McDermid, Patrick Prosser, Philip Roscoe, Ildikó Schlotter, Jay Sethuraman and Colin Sng.

The BibTeX references were compiled using “BibTeX Manager”, a Java application designed and implemented by Mitesh Furia under my

supervision as part of his development project for the MSc in Information Technology in academic session 2008-09. This very useful program helped me greatly to manage a bibliography of the size and complexity that emerged through the writing of this book.

Also on the technical front, other software applications that I have greatly benefited from include MiKTeX, TeXnicCenter and Dropbox. The latter allowed me to seamlessly transfer between my work PC, laptop and home PC in order to work on the book, without the need to copy files using email or pen drive.

During the period of the writing my research was financially supported by the Engineering and Physical Sciences Research Council, through grant EP/E011993/1 (‘‘MATCH-UP: Matching Under Preferences — Algorithms and Complexity’’, running from 1 June 2007 to 30 June 2010).

I am grateful to Scott Barr, and other friends in Linlithgow, who have spurred me on by asking ‘‘how’s the book?’’ at regular intervals.

My father Colin, himself an author of many books (on fantasy literature), has been a source of inspiration, as was my late mother Evelyn who died in 2003.

But lastly, and most importantly, I would like to thank my wife Carolynn for her love, understanding, support and encouragement through all aspects of this project. This book is dedicated to her, and to our children Susan and Paul.

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